FUZZY ACTIONS AS FUZZY GROUPOIDS

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ABSTRACT: *The purpose of this short note is to introduce a notion of T-fuzzy groupoid fuzzifying not only the set but also the groupoid operations (the partially defined multiplication as well as the inversion) and to associate to a fuzzy action in the sense [D. Boixader and J. Recasens, Fuzzy Sets and Systems, 2018] such a fuzzy groupoid.*

KEY WORDS: *groupoid, fuzzy action, fuzzy equivalence relation*

1. TERMINOLOGY AND NOTATION

The notion of (crisp) groupoid (also known as Brandt groupoid [2]) used in this paper is a generalization of the notion of group obtained by replacing the binary operation with a *partially defined* binary operation that is associative and has inverses and identities. More precisely, by a (crisp) groupoid we mean a set G, together with a subset $G^{(2)} \subset G \times G$, and two maps: a partially defined multiplication $m: G^{(2)} \to G$, and an inverse map $i: G \to G$ satisfying the following properties:

1. Inverses: $i(i(x)) = x$
2. Associativity: If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy, z), (x,yz) \in G^{(2)}$ and $m(m(x,y),z) = m(x,m(y,z))$.
3. Identities: for all $x \in G$, $m(x, i(x)) \in G^{(2)}$, $(i(x), x) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$ (respectively, $(z,x) \in G^{(2)}$), then $y = m(i(x),m(x,y))$ (respectively, $z = m(m(z,x),i(x))$.

Usually we write xy for $m(x,y)$, $(x,y) \in G^{(2)}$ and x^{-1} for $i(x)$, $x \in G$. The maps r and d on G, defined by $r(x) = xx^{-1}$ and respectively, $d(x) = x^{-1}x$, are called the range and respectively the domain map. It is easy to prove that $(x,y) \in G^{(2)}$ if and only if $d(x)=r(y)$. Also r and d have a common image $r(G)=d(G)$ called the unit space of G, which is denoted $G^{(0)}$. Hence, a groupoid can be viewed as à small category with inverses.

The concept of fuzzy set was introduced by Zadeh [15] starting from the observation: "more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership". Since then it have been used to model uncertain information in various areas: pattern recognition, computer vision, medical diagnosis, control, etc. Afterwards, the theory of fuzzy set was extended to other algebraic structure: groups ([11], [5]), equivalence relation ([4-6], [8-10], [12]), etc. In [3] we proposed a unifying approach to these fuzzy structures through the (Brandt) groupoids. If $I=[0,1]$ (or more generally, a complete lattice) and $T: I \times I \to I$ is a function, then a T-fuzzy subgroupoid (in the sense of [3]) of a groupoid G is function $\gamma : G \to I$ (i.e. a fuzzy subset on G) satisfying the following conditions:

1. $\gamma(xy) \geq T(\gamma(x),\gamma(y))$ for all $(x,y) \in G^{(2)}$.
2. $\gamma(x^{-1}) \geq \gamma(x)$ for all $x \in G$.
3. $\gamma(r(x)) \geq \gamma(x)$ for all $x \in G$.

We remark that in the above definition of fuzzy subgroupoid only the subset is fuzzy while the groupoid operations (partial multiplication and inversion) remain crisp. In this paper we try to fuzzify not only the subset but also the groupoid operations. Our approach differs to the one proposed in [14] (where the term groupoid refers to a set closed under a binary operation without any other further properties) and also is rather different to the one used in [13]. In the last section of this paper we show that the fuzzy actions introduced in [1] can be treated in the framework of fuzzy groupoids.

In this paper I = [0,1] (or more generally, a complete lattice) and $T: I \times I \to I$ is a fixed t-norm, i.e. a function $T: I \times I \to I$ which satisfies the following properties:

1. $T(a, b) = T(b, a)$ for all $a,b\in I$;
2. $T(a, b) \le T(c, d)$ if $a \le c$ and $b \le d$;
3. $T(a, T(b, c)) = T(T(a, b), c)$ for all $a,b,c\in I$;
4. $T(a, 1) = a$ for all $a\in I$

(see [7] for various examples of t-norms).

For the lattice I we denote the least upper bound by sup, the greatest lower bound by inf and we write

$$a\wedge b=T(a,b).$$

2. FUZZY GROUPOIDS

We propose a notion of fuzzy groupoid fuzzifying the groupoid operations.

Definition: A fuzzy groupoid is a nonempty set G together with two functions,

$$m:G\times G\to I \text{ and } i:G\to G,$$

satisfying the following conditions:

1. $m(x,y,a)\wedge m(a,z,c)\wedge m(y,z,b) \le m(x,b,c)$ for all $x,y,z,a,b,c\in G$.
2. $m(y,z,b)\wedge m(x,b,c)\wedge m(x,y,a) \le m(a,z,c)$ for all $x,y,z,a,b,c\in G$.
3. For every $x,y\in G$ we have
$$i(x,y)\le\gamma^{(2)}(x,y) \text{ and } i(y,x)\le \gamma^{(2)}(x,y),$$
where
$$\gamma^{(2)}(x,y)=\sup\{m(x,y,z),\ z\in G\}$$
4. $\gamma^{(2)}(y,x)\wedge i(y,a)\wedge m(a,y,u)\le m(u,x,x)$ for all $x,y,a,u\in G$.
5. $\gamma^{(2)}(x,y)\wedge i(y,a)\wedge m(y,a,v)\le m(x,v,x)$ for all $x,y,a,v\in G$.

For a fuzzy groupoid G(m,i) we write

$$\gamma(x)=\sup\{i(x,y)\wedge i(y,x),\ x,y\in G\}$$
$$\gamma^{(2)}(x,y)=\sup\{m(x,y,z),\ z\in G\}$$
$$\gamma^{(0)}(u)=m(u,u,u)\wedge i(u,u)\wedge i(u,u)$$
$$r(x,u) = \sup\{i(x,y)\wedge m(x,y,u)\wedge\gamma^{(0)}(u),\ y\in G\}$$
$$d(x,u) = \sup\{i(x,y)\wedge m(y,x,u) \wedge\gamma^{(0)}(u),,\ y\in G\}$$

In the previous definition m(x,y,z) can be viewed as the degree to which z=xy and i(x,y) can be interpreted as the degree to which $y=x^{-1}$. Conditions 1 and 2 fuzzify the properties

$$(xy)z=c \Rightarrow x(yz)=c$$
$$x(yz)=c \Rightarrow (xy)z=c$$

Condition 3 fuzzyfies the properties:

$$(x,x^{-1})\in G^{(2)} \text{ and } (x^{-1}, x)\in G^{(2)}$$

Condition 4 fuzzyfies the property

$$(y,x) \in G^{(2)} \Rightarrow (y^{-1}y)x=x$$

and condition 4 fuzzyfies the property

$$(x,y) \in G^{(2)} \Rightarrow x(yy^{-1})=x$$

If G(m,i) is a fuzzy groupoid in the preceding sense, then

1. $\gamma^{(0)}(u)\le \gamma(u)$ for all $u\in G$.
2. $\gamma^{(0)}(u)\le \gamma^{(2)}(u,u)$ for all $u\in G$.

Indeed, for every $u\in G$, we have

$$\gamma^{(0)}(u)=m(u,u,u)\wedge i(u,u) \wedge i(u,u)$$

hence

$$\gamma^{(0)}(u)\le i(u,u)\wedge i(u,u)\le \sup\{i(u,y)\wedge i(y,u) ,y\in G\}= \gamma(u).$$

Moreover

$$\gamma^{(0)}(u)\le m(u,u,u)\le \sup\{m(u,u,z),\ z\in G\} = \gamma^{(2)}(u,u).$$

3. FUZZY ACTIONS

The purpose of this section is to construct a fuzzy groupoid (in the sense introduced in Section 2) associated to a fuzzy action (in the sense of [1, Definition 3.3])

If X is a set and Γ is a group with neutral element e, then a function $\alpha: G \times X \times X\to I$ fuzzy action of Γ on X in the sense of [1] if and only if for all $g, h \in \Gamma$, and all $x,y,z\in X$ we have

1. $\alpha(hg, x, y)\wedge\alpha(g, x, z)\le\alpha(h, z, y)$
2. $\alpha(g, x, z)\wedge\alpha(h, z, y)\le\alpha(hg, x, y)$
3. $\alpha(e,x,x) = 1$.

Let us consider an indistinguishability operator E on X (see [12] or [10]). That is a function $E:X\times X\to I$ such that for all $x, y, z\in X$ following conditions are satisfied:

1. $E(x, x) = 1$ (Reflexivity)
2. $E(x, y) = E(y, x)$ (Symmetry)
3. $E(x, y)\wedge E(y, z)) \le E(x, z)$ (Transitivity)

(in fact E is a particular fuzzy equivalence relation in the sense of [3]).

We assume that for all $g\in\Gamma$ and $x,y,z\in X$ we have the following compatibility condition between the indistinguishability operator E and the fuzzy action α:

$$\alpha(g,x,y)\wedge E(z,y)\le \alpha(g,x,z)$$
$$\alpha(g,x,y)\wedge E(z,y)\wedge E(y,u)\le \alpha(g,x,z)\wedge E(z,u).$$
$$\alpha(e,a,x)\wedge E(v,a)\le\alpha(e,v,x)\wedge E(x,v)$$

An example of such indistinguishability operator is

$$E(x,y)=\alpha(e,x,y) \text{ for all } x,y\in X$$

(Definition 3.9 [1] and Proposition 3.10 [1]), for a t-norm T having the property that

$T(a,a) = a$ for all $a\in I$.

In particular, for the t-norm $T=T_{min}$:

$T(\mu,\nu)=\mu\wedge\nu=\inf\{\mu,\nu\}$),

the indistinguishability operator defined in [1, Definition 3.9] satisfies the compatibility condition with the action α

Let us define a fuzzy groupoid G(m,i) associated to the fuzzy action α [1, Definition 3.3]), thus extending the construction of the transformation groupoid (associated to a crisp action α of a group Γ on a set X) to the fuzzy settings. As in the crisp case we take $G=\Gamma\times X$.

For $G=\Gamma\times X$. the fuzzy operations are defined in the following way:

$i: G\times G \to I$

$$i(g,x,h,y) = \begin{cases} \alpha(g,x,y) \text{ if } h=g^{-1} \\ 0, \text{ otherwise} \end{cases}$$

$m:G\times G\times G \to I$

$$m(g,x,h,y,k,z) = \begin{cases} \alpha(h,y,x)\wedge E(z,y), \text{ if } k=gh \\ 0, \text{ otherwise} \end{cases}$$

Let us note that if $g=h^{-1}$

$i(h,y,g,x) = \alpha(h,y,x)=\alpha(h^{-1},x,y)= \alpha(g,x,y)$, and if $g\neq h^{-1}$, $i(h,y,g,x) = 0$,. Thus

$i(g,x,h,y) = i(h,y,g,x)$

We have

$\gamma(g,x) =$
$= \sup\{ i(g,x,h,y)\wedge i(h,y,g,x), (h,y)\in G\}$
$= \sup\{ i(g,x,h,y)\wedge i(h,y,g,x), y\in X, h=g^{-1}\}$
$= \sup \{\alpha(g,x,y)\wedge\alpha(g^{-1},x,y), y\in X\}$
$= \sup\{\alpha(g,x,y)\wedge\alpha(g,x,y), y\in X\}$

We also have

$\gamma^{(2)}(g,x,h,y)=\sup\{m(g,x,h,y,k,z), (k,z)\in G\}$
$=\sup\{\alpha(h,y,x)\wedge E(z,y), z\in X\}$
$\ge \alpha(h,y,x)$ $(z=y)$

Om the other hand

$\gamma^{(2)}(g,x,h,y) = \sup\{\alpha(h,y,x)\wedge E(z,y), z\in X\}$
$\le \alpha(h,y,x)$.

Consequently,

$\gamma^{(2)}(g,x,h,y)= \alpha(h,y,x)$.

and

$\gamma^{(2)}(g,x,h,y)\ge \alpha(h,y,x)= i(h,y,g,x)= i(g,x,h,y)$.

Hence condition 3 from definition of a fuzzy groupoid in the sense introduced in Section 2 is verified.

Let us verify conditions 1 and 2:

$m(g,x,h,y,gh,a)\wedge m(gh,a,k,z,ghk,c)\wedge m(h,y,k,z,hk,b)$
$=\alpha(h,y,x)\wedge E(a,y)\wedge\alpha(k,z,a)\wedge E(z,c)\wedge\alpha(k,z,y)\wedge E(b,z)$
$=\alpha(h,y,x)\wedge\alpha(k,z,y) \wedge\alpha(k,z,a)\wedge E(a,y) \wedge E(z,c) \wedge E(b,z)$
$\le\alpha(hk,z,x)\wedge E(b,z)\wedge\alpha(k,z,a)\wedge E(a,y) \wedge E(z,c)$
$\le\alpha(hk,z,x)\wedge E(b,z) \wedge E(z,c)$
$\le \alpha(hk,b,x)\wedge E(c,b)$
$=m(g, x, hk, b, ghk, c)$

$m(y,z,b)\wedge m(x,b,c)\wedge m(x,y,a) \le m(a,z,c)$
$m(h,y,k,z,hk,b)\wedge m(g,x,hk,b, ghk,c)\wedge m(g,x,h,y,gh,a)$
$=\alpha(k,z,y)\wedge E(b,z)\wedge\alpha(hk,b,x)\wedge E(c,b)\wedge\alpha(h,y,x) \wedge E(a,y)$
$=\alpha(k,z,y)\wedge E(a,y)\wedge E(c,b)\wedge E(b,z)\wedge\alpha(hk,b,x)\wedge \alpha(h,y,x)$
$\le\alpha(k,z,y)\wedge E(a,y)\wedge E(c,b)\wedge E(b,z)$
$\le\alpha(k,z,y)\wedge E(c,b)\wedge E(b,z)$
$\le \alpha(k,z,a)\wedge E(c,z)$
$= m(gh,a,k,z,ghk,c)$

Finally let us verify conditions 4 and 5 from the definition of fuzzy groupoid (in Section 2):

$\gamma^{(2)}(h,y, g,x)\wedge i(h,y,h^{-1},a)\wedge m(h^{-1},a,h,y,e,u)=$

$=\alpha(g,x,y)\wedge i(h,y,h^{-1},a)\wedge m(h^{-1},a,h,y,e,u)$
$= \alpha(g,x,y)\wedge\alpha(h,y,a)\wedge\alpha(h,y,a)\wedge E(u,y)$
$\leq\alpha(g,x,y)\wedge E(u,y)$
$\leq \alpha(g,x,u)$
$=\alpha(g,x,u)\wedge E(x,x)$
$=m(e,u,g,x,g,x)$

$\gamma^{(2)}(g,x, h,y)\wedge i(h,y,h^{-1},a)\wedge m(h, y, h^{-1}, a,,e,v)$
$=\alpha(h,y,x)\wedge i(h,y,h^{-1},a)\wedge m(h, y, h^{-1}, a,,e,v)$
$= \alpha(h,y,x)\wedge\alpha(h,y,a)\wedge\alpha(h^{-1},a,y)\wedge E(v,a)$
$= \alpha(h,y,x)\wedge\alpha(h^{-1},a,y)\wedge\alpha(h,y,a)\wedge E(v,a)$
$\leq\alpha(e,a,x) \wedge\alpha(h,y,a)\wedge E(v,a)$
$\leq\alpha(e,a,x)\wedge E(v,a)$
$\leq\alpha(e,v,x)\wedge E(x,v)$
$=m(g,x,e,v,g,x)$

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