APPROXIMATING SOLUTIONS FOR A CLASS OF STOCHASTIC FRACTIONAL LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS

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ABSTRACT: *In this paper we consider a linear quadratic control problem for a class of discrete-time fractional order systems with multiplicative noise and we find lower bounds of the optimal cost and approximating solutions for the optimal control law. Our approach is based on the Riccati equation method used in [7,8]. Unlike [7,8], in this paper the memory of the system also refers to the control sequence and not only to the state variable.*

KEY WORDS: *discrete-time Riccati equation of control, optimal control, stochastic equations*

1. INTRODUCTION

Linear quadratic optimal control of linear fractional order systems (LFOS) is a new an important issue in control theory. For more details we refer the reader to [1-2],[6] and the references therein. Using a linear form of the fractional system (similar to the ones in [7]) and the classical theory based on Riccati equations, we find a lower bound of the optimal control problem under discussion [2].

2. PRELIMINARIES

Let $\alpha \in (0,2)$ and $h > 0$ be fixed. For any $j \in \mathbf{N}$, $\binom{\alpha}{j}$ denotes the generalized binomial coefficient defined by

$$\binom{\alpha}{j} = \begin{cases} 1, j = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha+1-j)}{j!}, j \in \mathbf{N}^* \end{cases}$$

and

$$\Delta^{[\alpha]} x_{k+1} = \frac{1}{h^\alpha} \sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} x_{k+1-j}, h > 0$$

is the discrete-time version of the Grünwald-Letnikov fractional order derivative (see for e.g. [3-4]).

Let $\{\xi_k\}_{k \in \mathbf{N}}$ be a sequence of real-valued, mutually independent random variables on the probability space $(\Omega, \mathbf{G}, P)$ having the property that $E[\xi_k] = 0, E[\xi_k^2] = 1, k \in \mathbf{N}$. where $E[\xi]$ is the mean (expectation) of ξ_k. Let $\mathbf{G}_n$ be the $\sigma-$ algebra generated by $\{\xi_i, 0 \le i \le n-1\}, n \in \mathbf{N}^*$.

We consider the stochastic discrete-time fractional system with control

$$\Delta^{[\alpha]} x_{k+1} = \mathrm{A} x_k + \sum_{i=0}^{k} \xi_i \mathrm{B}_i x_i + \sum_{i=0}^{k} \mathrm{D}_i u_{k-i} + \sum_{i=0}^{k} \xi_i \mathrm{F}_i u_{k-i}, k \in \mathrm{N}$$

$$x_0 = x \in \mathrm{R}^d,$$

where A, $\mathrm{B}_i \in \mathrm{R}^{d \times d}$, D_i, $\mathrm{F}_i \in \mathrm{R}^{d \times m}, i \in \mathrm{N}$ and the control $u = \{u_i\}_{i \in \mathrm{N}}$ belongs to a class of admissible controls $\mathbf{U}^a$ formed by all sequences u which elements u_k are R^m-valued and $\mathbf{G}_k$ - measurable random variables, which satisfy the condition $E\left[\|u_k\|^2\right] < \infty$ for all $k \in \mathrm{N}$.

The above system can be equivalently rewritten as

$$x_{k+1} = \sum_{j=0}^{k} \left(A_j x_{k-j} + \xi_j B_j x_{k-j}\right) +$$

$$\sum_{j=0}^{k} \left(D_j u_{k-j} + \xi_j F_j u_{k-j}\right), \tag{2.1}$$

$$x_0 = x \in \mathrm{R}^d,$$

where $A_0 = h^\alpha \mathsf{A} + \alpha I_{\mathrm{R}^d}$, $A_i = c_i I_{\mathrm{R}^d}$ $c_i := (-1)^i \binom{\alpha}{i+1}$ and $T_i = h^\alpha \mathsf{T}_i$, for any $T_i = B_i$, D_i , F_i , $\mathsf{T}_i = \mathsf{B}_i$, D_i , F_i , $, i \in \mathrm{N}$.

Our goal is to find an approximating solution for the following optimal control problem

O: Let $x_0 \in \mathrm{R}^d$ and $N \in \mathrm{N}$ be fixed and let $C_j \in \mathrm{R}^{p\times d}, S \in \mathrm{R}^{d\times d}, S \geq 0,$ $K_j \in \mathrm{R}^{m\times m}, K_j > 0$.

Minimize the cost functional

$$(2.2) \quad I_{x_0,N}(u) = E < Sx_N, x_N > + \sum_{n=0}^{N-1} E\left[\left(\|C_n x_n\|^2 + < K_n u_n, u_n >\right)\right]$$

subject to (2.1), over the class $\mathbf{U}^a_{0,N-1} \subseteq \mathbf{U}^a_1$ of segments u = { u0, u1, u2,..,uN-1, 0, 0,..} of admissible controls.

3. AN EQUIVALENT LINEAR FORM OF THE SYSTEM WITH MEMORY

Let $\mathbf{A},\mathbf{B} : (\mathrm{R}^d)^N \to (\mathrm{R}^d)^N$ be the linear operators defined by the matrices

$$\mathbf{A} = \begin{pmatrix} A_0 & c_1 I_{\mathrm{R}^d} & \dots & c_{N-1} I_{\mathrm{R}^d} \\ I_{\mathrm{R}^d} & 0 & . & 0 \\ . & I_{\mathrm{R}^d} & . & . \\ . & . & I_{\mathrm{R}^d} & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} B_0 & B_1 & . & . & B_{N-1} \\ 0 & 0 & . & . & 0 \\ . & . & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & 0 \end{pmatrix}.$$

We also introduce the following N×N matrices of $d\times m$ matrices:

$$\mathbf{D_k} = \begin{pmatrix} D_0 & . & D_k & 0 & 0 \\ 0 & 0 & . & . & 0 \\ . & . & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & 0 \end{pmatrix}.$$

$$\mathbf{F_k} = \begin{pmatrix} F_0 & 0 & F_k & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & 0 \end{pmatrix}.$$ Also for all

$(v_0, v_1, ..., v_{N-1}) \in (\mathrm{R}^m)^N$

$$\mathbf{K_k}(v_0, v_1, ..., v_{N-1}) = (K_k v_0, ..., 0, ..., 0) \in (\mathrm{R}^m)^N$$

Similarly, for all $(v_0, v_1, ..., v_{N-1}) \in (\mathrm{R}^d)^N$ we introduce the linear operators

$$\mathbf{C_k}(v_0, v_1, ..., v_{N-1}) = (C_k v_0, 0, ..., 0)$$

$$\mathbf{S}(v_0, ..., v_{N-1}) = (S v_0, 0, ..., 0).$$

We note that the new matrices satisfy the condition $\mathbf{K}_k, \mathbf{S} \geq 0$. Let $x_0, x_1, ..., x_k, ...$ be a solution of (2.1). For any $k < N, X_k^T = (x_k, x_{k-1}, ..., x_0, 0, .., \underset{N}{0}) \in (\mathrm{R}^d)^N$ is a solution of the linear system

$$(3.1) \quad \begin{aligned} X_{k+1} &= \mathbf{A}X_k + \xi_k \mathbf{B} X_k + \mathbf{D}_k U_k + \xi_k \mathbf{F}_k U_k, \\ X_0 &= (x_0, 0, ..., \underset{N}{0}), \end{aligned}$$

where the control $U = \{U_k\}_{k\in\mathrm{N}} \subset (\mathrm{R}^m)^N$ belongs to the set U^a of admissible controls sequences $\{U_k\}_{k\in\mathrm{N}}$ having the property that $U_k = (u_k, u_{k-1}, ..., u_0, 0, ..\underset{N}{0})$ are $(\mathrm{R}^m)^N$ -valued, $\mathbf{G}_k$ -measurable random variables and $E\left[\|U_k\|^2\right] < \infty$ for all $k \in \mathrm{N}$. It is not difficult to see that for this class of admissible controls (2.1) is an equivalent form of (3.1). Since

$$\mathbf{C_k} X_k = \mathbf{C_k}(x_k, x_{k-1}, ..., x_0, 0, ..0) = (C_k x_k, 0, ..., 0),$$

We define the operator $\mathbf{K}$ by

$$\mathbf{K}U_{N-1} = \mathbf{K}_k(u_{N-1}, ..., ..\underset{N}{u_0}) = (K_{N-1}u_{N-1}, ..., K_0 u_k)$$

And the cost functional (2.2) can be equivalently rewritten as

$$(3.2) \quad I_{x_0,N}(U) = E[\sum_{k=0}^{N-1} \left\langle \mathbf{C_k}^* \mathbf{C_k} X_k, X_k \right\rangle + + \left\langle \mathbf{K}U_{N-1}, U_{N-1} \right\rangle + \left\langle \mathbf{S}X_N, X_N \right\rangle].$$

Substituting X_N given by (3.1) in (3.2), we get $I_{x_0,N}(U)=\sum_{n=0}^{N-2}E[\langle \mathbf{C_n}^*\mathbf{C_n}X_n,X_n\rangle]$
$+E[\langle(\mathbf{C_{N-1}}^*\mathbf{C_{N-1}}+\mathbf{A}^*\mathbf{SA}+\mathbf{B}^*\mathbf{SB})X_{N-1},X_{N-1}\rangle$
$+2\langle(\mathbf{D}^*_{N-1}\mathbf{SA}+\mathbf{F}^*_{N-1}\mathbf{SB})X_{N-1},U_{N-1}\rangle+$
$\langle(\mathbf{K}+\mathbf{D}^*_{N-1}\mathbf{SD}_{N-1}+\mathbf{F}^*_{N-1}\mathbf{SF}_{N-1})U_{N-1},U_{N-1}\rangle].$

Now let $\mathsf{U}^a_{0,N-1}$ be the class of all finite segments $U_0,...,U_{N-1}$ of sequences $U\in\mathsf{U}^a$. Then the optimal control problem O is equivalent with a new minimal optimal control problem $\mathbf{O}_1$ defined by system (3.1), $I_{x_0,N}(U)$ and $\mathsf{U}^a_{0,N-1}$. The proof is very similar to the one in [7]. Indeed, for any $u\in\mathbf{U}^a_{0,N-1}$, the segment $U=\{U_k=(u_k,...,u_0,0,..,0),k=0,..,N-1\}$ belongs to $\mathsf{U}^a_{0,N-1}$ and $I_{x_0,N}(U)=I_{x_0,N}(u)$. Conversely, given $U\in\mathsf{U}^a_{0,N-1}$, we define $u=U_{N-1}$, Obviously, $u\in\mathbf{U}^a_{0,N-1}$ and $I_{x_0,N}(U)=I_{x_0,N}(u)$. It follows that $\overline{U}$ is optimal for $I_{x_0,N}(U)$ if and only if $\overline{u}=\{\overline{u}_k=\overline{U}_{N-1k},k=0,..,N-1\}$ is optimal for $I_{x_0,N}(u)$ and $I_{x_0,N}(\overline{U})=I_{x_0,N}(\overline{u})$. We just have reduced our optimal control problem to a linear quadratic optimal control problem $\mathbf{O}_1$ for linear stochastic system (3.1). Since, all the coefficients $\mathbf{K^k}$ of the term $\sum_{n=0}^{N-2}\langle\mathbf{K^k}U_k,U_k\rangle$ from the cost functional $I_{x_0,N}(U)$ are null it follows that the new form of the optimal control does not satisfy the positivity condition $\mathbf{K^k}>0$, $k=0,..,N-1$, required by the classical method based on Riccati equations (see [2] for e.g.) and we cannot solve $\mathbf{O}_1$ by using this method (see [2], [3]).

As in [7] we introduce the following new optimal cost

$$I_{x_0,N,\varepsilon}(U)=\sum_{k=0}^{N-2}(E[\langle\mathbf{C_k}^*\mathbf{C_k}X_k,X_k\rangle]+$$
$$E[\langle\mathsf{K}^{\varepsilon}_kU_k,U_k\rangle])+\langle\mathsf{K}_{N-1}U_{N-1},U_{N-1}\rangle+$$
$$E[\langle(\mathbf{C_{N-1}}^*\mathbf{C_{N-1}}+\mathbf{A}^*\mathbf{SA}+\mathbf{B}^*\mathbf{SB})X_{N-1},X_{N-1}\rangle+$$
$$2\langle(\mathbf{D}^*_{N-1}\mathbf{SA}+\mathbf{F}^*_{N-1}\mathbf{SB})X_{N-1},U_{N-1}\rangle$$

where $\varepsilon>0$ is fixed and such that

(3.3) $K_k>N\varepsilon$, for all $k=0,..,N$,

$\mathbf{K_{mod}}=\mathrm{diag}(K_{N-1},\frac{N-1}{N}K_{N-2}...,\frac{2}{N}K_1,\frac{1}{N}K_0)$,

$\mathsf{K}^{\mathrm{mod},\varepsilon}_k=\mathrm{diag}(\frac{1}{N}K_k-\mathbf{I_N}\varepsilon,..,\frac{1}{N}K_0-\mathbf{I_N}\varepsilon,....,$
$\frac{\varepsilon}{N}\mathbf{I_N},...,\frac{\varepsilon}{N}\mathbf{I_N})$,

$\mathsf{K}_{N-1}=\mathbf{K_{mod}}+\mathbf{D}^*_{N-1}\mathbf{SD}_{N-1}+\mathbf{F}^*_{N-1}\mathbf{SF}_{N-1}>0$

and $\mathbf{I_N}$ is the identity operator on $(\mathrm{R}^m)^N$.

Now, it is not difficult to see that

$$(3.4)\sum_{n=0}^{N-2}\langle\mathsf{K}^{\mathrm{mod},\varepsilon}_kU_k,U_k\rangle+\langle\mathbf{K_{mod}}U_{N-1},U_{N-1}\rangle=$$
$$\langle\mathbf{K}U_{N-1},U_{N-1}\rangle-\varepsilon\sum_{n=0}^{N-2}(1-k(N+1)/N)\langle u_k,u_k\rangle$$

and

$$I_{x_0,N,\varepsilon}(U)=I_{x_0,N}(U)$$
$$-\varepsilon\sum_{n=0}^{N-2}(1-k(N+1)/N)\langle U_k,U_k\rangle.$$

Moreover, condition (3.3) ensures that $\mathsf{K}^{\mathrm{mod},\varepsilon}_k>0$ for all $k=0,..,N-2$.

Let us consider the class of admissible controls $\overline{\mathsf{U}}^a_{0,N-1}$ formed by all control sequences $\{U_k\}_{k\in\{0,..,N-1\}}$ having the property that U_k, $k\in\mathrm{N}$ is a $(\mathrm{R}^m)^N$ -valued, $\mathbf{G}_k$ -measurable random variable such that $E[\|U_k\|^2]<\infty$ and such that its last *N-k* components are null. Obviously, $\overline{\mathsf{U}}^a_{0,N-1}\supset\mathsf{U}^a_{0,N-1}$, Since $\mathbf{K_{mod}}$ is also a positive definite matrix, it follows that the problem of minimizing this new cost function over the class of admissible controls $\overline{\mathsf{U}}^a_{0,N-1}$ defines a new optimal control problem $\mathbf{O}_\varepsilon$, which satisfies the requirement that the weighting coefficients of the control variable are positive. Therefore we can apply classical results from [2] to solve $\mathbf{O}_\varepsilon$. However, $\overline{\mathsf{U}}^a_{0,N-1}\neq\mathsf{U}^a_{0,N-1}$ which implies that the optimal control $\overline{U}$ of $\mathbf{O}_{\varepsilon}$, is not necessarily optimal for problem $\mathbf{O}_1$ and consequently of for problem problem O.

4. PERTURBED RICCATI EQUATION OF CONTROL

The backward discrete-time Riccati equation associated with $\mathbf{O}_{\varepsilon}$ is the following

$$R_n^{\varepsilon} = \mathbf{A}^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{B}^* R_{n+1}^{\varepsilon}\mathbf{B} + \mathbf{C_n}^*\mathbf{C_n} - \left(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{B}\right)^* \cdot (\mathsf{K}_n^{\mathrm{mod},\varepsilon} + \mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{F}_n)^{-1}$$

$$(4.1)\quad \left(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{B}\right), n < N-1$$

$$R_{N-1}^{\varepsilon} = \mathbf{C}_{N-1}{}^*\mathbf{C}_{N-1} + \mathbf{A}^*\mathbf{SA} + \mathbf{B}^*\mathbf{SB} - \left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right)^* \cdot \left(\mathsf{K}_{N-1}\right)^{-1} \left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right).$$

An easy computation and formula (4.8) from [6] shows that

$$R_n^{\varepsilon} = \left(\mathbf{A} + \mathbf{D}_n W_n\right)^* R_{n+1}^{\varepsilon}\left(\mathbf{A} + \mathbf{D}_n W_n\right) + \left(B + \mathbf{F}_n W_n\right)^* R_{n+1}^{\varepsilon}\left(B + \mathbf{F}_n W_n\right) + C_n^* C_{\mathbf{n}} + \varepsilon W_n^* W_n, n = 0,.., N-2.$$

$$R_{N-1}^{\varepsilon} = C_{N-1}^* C_{N-1} + \left(\mathbf{A} + \mathbf{D}_{N-1}W_{N-1}\right)^* \mathbf{S} \bullet \left(\mathbf{A} + \mathbf{D}_{N-1}W_{N-1}\right) + \left(\mathbf{B} + \mathbf{F}_{N-1}W_{N-1}\right)^* \mathbf{S}\left(\mathbf{B} + \mathbf{F}_{N-1}W_{N-1}\right) + W_{N-1}^*\left(\mathbf{K}_{N-1}\right)W_{N-1},$$

for

$$(4.2)\, W_{N-1} = -\left(\mathsf{K}_{N-1}\right)^{-1}\left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right),$$

And

$$(4.3)\quad W_n = -(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{F}_n + \mathsf{K}_n^{\mathrm{mod},\varepsilon})^{-1} \cdot \left(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{B}\right), n = 0,.., N-2.$$

Now it is clear that $R_n^{\varepsilon} \geq 0, n = 0,.., N-1$.

Arguing as in [2], we can prove the following lemma.

Lemma 1 *The cost functional* $I_{x_0,N,\varepsilon}(U)$ *can be equivalently rewritten as*

$$I_{x_0,N,\varepsilon}(U) = E\left[\left\langle R_0^{\varepsilon} X_0, X_0\right\rangle\right] + \left\langle \mathsf{K}_{N-1}\left(W_{N-1}X_{N-1} - U_{N-1}\right), \left(W_{N-1}X_{N-1} - U_{N-1}\right)\right\rangle + \sum_{k=0}^{N-2} E\left[\left\|(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{F}_n + \varepsilon \mathbf{I}_N)^{1/2}\left(W_n X_n - U_n\right)\right\|^2\right],$$

where R_n^{ε} *is the unique solution of (4.1).*

5. MAIN RESULTS

In this section we first prove that optimal control problem $\mathbf{O}_{\varepsilon}$ has a solution for any ε belonging to an appropriate neighbour of 0. Using this result we then solve problem O by showing that the optimal control gain can be approximated by using solutions of problem $\mathbf{O}_{\varepsilon}$ and $\mathbf{O}_{-\varepsilon}$.

Theorem 1. *a) Let* $\varepsilon > 0$ *satisfy condition (3.3). If* $\{R_n^{\varepsilon}\}_{n=0,..,N-1}$ *is the unique solution of the Riccati equation (4.1) and* $W_n, n = 0,.., N-1$ *are defined by (4.2), (4.3), then* $\mathbf{O}_{\varepsilon}$ *admits an optimal control gain*

$$(5.1)\, \overline{U} = \{\overline{U}_0 = W_0 X_0,.., \overline{U}_n = W_n X_n, ..., \overline{U}_{N-1} = W_{N-1}X_{N-1}\}$$

from $\overline{\mathsf{U}}_{0,N-1}^{a}$ *which minimizes the cost functional* $I_{x_0,N,\varepsilon}(U)$ *and*

$$(5.2)\ \min_{U \in \mathsf{U}_{0,N-1}^{a}} I_{x_0,N,\varepsilon}(U) = E\left[\left\langle R_0^{\varepsilon} X_0, X_0\right\rangle\right].$$

Proof. The existence of the optimal control $\overline{U}$ given by (5.1) is a well know result and its proof is omitted [2]. We only note that the special form of the Riccati equation (4.1) ensures that the last *N-n* components of $\overline{U}_n$ are null, which implies that $\overline{U} \in \overline{\mathsf{U}}_{0,N-1}^{a}$.

Remark. 1. The above conclusion implies that the last *N-n* components of the diagonal of $\mathsf{K}_n^{\mathrm{mod},\varepsilon}$=diag($\frac{1}{N}K_n - \mathbf{I_N}\varepsilon,.., \frac{1}{N}K_0 - \mathbf{I_N}\varepsilon,....,$ $\frac{\varepsilon}{N}\mathbf{I_N},.., \frac{\varepsilon}{N}\mathbf{I_N}$), does not influence the value of the optimal control. It follows that if we replace ε by $-\varepsilon$ in the above theorem its conclusions remain valid for any negative $-\varepsilon$.

Lemma 2. *Let a, b, a ≤ b be* ***real number which satisfy condition (3.3) (with*** ε *replaced by a or/and b) if they* ***are positive. If*** $\{R_n^{a}\}_{n=0,..,N-1}, \{R_n^{a}\}_{n=0,..,N-1}$ *are the unique solutions of the Riccati equation (4.1)(obtained by replacing* ε *by -a or –b when a and b are negative) then*

(5.3) $0 \le \langle R_0^a X_0, X_0 \rangle \le \langle R_0^b X_0, X_0 \rangle$

for all $X_0^T = (x_0, \underbrace{0,..,0}_{N}) \in (\mathrm{R}^d)^N$.

Proof. Relation (3.4) shows that, for any $a \le b$ (satisfing condition (3.3) if they are positive) we have

$0 \le I_{x_0,N,a}(U) \le I_{x_0,N,b}(U)$ and

(5.4) $\min_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N,a}(U) \le \min_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N,b}(U)$.

From Theorem 1 and Remark 1, it follows that $\min_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N,\varepsilon}(U) = \langle R_0^\varepsilon X_0, X_0 \rangle \ge 0$ for any ε (negative or positive) and we obtain (5.3).

The following theorem provides a lower bound for the optimal cost of the problem O.

Theorem 2.

$\inf_{u \in \mathrm{U}^a} I_{x_0,N}(u) \ge \lim_{\substack{b \to 0 \\ b<0}} \langle R_0^b X_0, X_0 \rangle$.

Proof. From (3.4) it follows that there is an $U \in \mathrm{U}_{0,N-1}^a$ and b < 0 such that $I_{x_0,N}(u) \ge I_{x_0,N,b}(U)$ which implies that

(5.5) $\inf_{u \in \mathrm{U}^a} I_{x_0,N}(u) \ge \min_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N,b}(U) = \langle R_0^b X_0, X_0 \rangle$

for $X_0^T = (x_0, \underbrace{0,..,0}_{N}) \in (\mathrm{R}^d)^N$.

Taking into account Lemma 2 and passing to the limit as $b \to 0$ in (5.5) we see that $\inf I_{x_0,N}(u) \ge im_{\substack{b \to 0 \\ b<0}} \langle R_0^b X_0, X_0 \rangle$.

The following theorem provides an upper bound of $\inf_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N}(u)$.

Theorem 3. *If there is a positive real number b such that the optimal control problem* O_b *has a solution from* $\mathrm{U}_{0,N-1}^a$ *then*

$\lim_{a \to 0, a<0} \langle R_0^a X_0, X_0 \rangle \le \inf_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N}(u)$

$\le \langle R_0^b X_0, X_0 \rangle$.

Proof. Let $U \in \mathrm{U}_{0,N-1}^a$ be the optimal control law of problem O_b. We know that there is a $u \in \mathrm{U}^a$ defined from U such $I_{x_0,N}(u) \le I_{x_0,N,b}(U) = \langle R_0^b X_0, X_0 \rangle$. It follows that $\inf_{u \in \mathrm{U}^a} I_{x_0,N}(u) \le \langle R_0^b X_0, X_0 \rangle$. From Lemma 2, we know that $\langle R_0^a X_0, X_0 \rangle \le \langle R_0^b X_0, X_0 \rangle$ for any *a < 0*. Since $I_{x_0,N,a}(U) \le I_{x_0,N}(u)$ (by (3.4)), we deduce that $\langle R_0^a X_0, X_0 \rangle \le \inf_{u \in \mathrm{U}^a} I_{x_0,N}(u) \le \langle R_0^b X_0, X_0 \rangle$. The conclusion follows by passing to the limit in the above inequality. Moreover, the optimal control *U* provides an approximation of the optimal cost of problem O with an error less than the difference $\langle R_0^b X_0, X_0 \rangle$ - $\lim_{b \to 0} \langle R_0^{-b} X_0, X_0 \rangle$.

6. CONCLUSION

A lower bound for the quadratic optimal cost of the optimal control problem (2.1) were found by using a linear form of the fractional discrete-time system and the classical method based on Riccati equations. As a consequence, some sufficient conditions for the existence of an approximating optimal control were given. We hope that these results will help us to find better estimations of $\inf_{U \in \mathrm{U}_{0,N-1}^a} I_{x_0,N}(u)$ and of the corresponding control law.

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