MATLAB SOLUTIONS FOR DISCRETE-TIME RICCATI EQUATIONS OF STOCHASTIC FRACTIONAL LINEAR QUADRATIC OPTIMAL CONTROL AND APPLICATIONS

Ungureanu Viorica Mariela, Associate Professor, *"Constantin Brâncuși" University from Târgu Jiu, ROMANIA*

ABSTRACT: *In this paper we study solution properties for a class of discrete-time Riccati equations of stochastic control associated to discrete-time fractional order systems with control and multiplicative white noise. We provide MATLAB simulations of the asymptotic behaviour of these solutions and graphic representations of the lower bounds of the quadratic cost functional to be minimized.*

KEY WORDS: *discrete-time Riccati equation of control, optimal control, stochastic equations*

1. INTRODUCTION

Riccati equations of stochastic control have attracted the interest of the scientists for a long time (see [1-2],[6] and the references therein) due to their important role in control theory. In this paper we study the asymptotic properties of their solutions by using MATLAB simulations. The results are used to obtain lower bounds for a linear quadratic cost functional associated with a linear fractional order system of Grünwald-Letnikov type.

2.RICCATI EQUATIONs OF CONTROL

Let us consider the following discrete-time Riccati equation (see [7])

$$R_n^{\varepsilon} = \mathbf{A}^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{B}^* R_{n+1}^{\varepsilon}\mathbf{B} + \mathbf{C}_n{}^*\mathbf{C}_n - \left(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{B}\right)^* \cdot \left(\mathsf{K}_n^{\mathrm{mod},\varepsilon} + \mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{F}_n\right)^{-1}$$

$$\left(\mathbf{D}_n^* R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^* R_{n+1}^{\varepsilon}\mathbf{B}\right), n < N-1 \tag{2.1}$$

$$R_{N-1}^{\varepsilon} = \mathbf{C}_{N-1}{}^*\mathbf{C}_{N-1} + \mathbf{A}^*\mathbf{SA} + \mathbf{B}^*\mathbf{SB} - \left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right)^* \cdot \left(\mathsf{K}_{N-1}\right)^{-1} \left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right).$$

where $c_i := (-1)^i \binom{\alpha}{i+1}, i \in \mathbf{Z},$

$$\mathbf{A} = \begin{pmatrix} A_0 & c_1 I_{\mathrm{R}^d} & \dots & c_{N-1} I_{\mathrm{R}^d} \\ I_{\mathrm{R}^d} & 0 & . & 0 \\ . & I_{\mathrm{R}^d} & . & . \\ . & . & I_{\mathrm{R}^d} & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} B_0 & B_1 & . & . & B_{N-1} \\ 0 & 0 & . & . & 0 \\ . & . & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & & 0 \end{pmatrix}$$ are N×N blocks of d×d real matrices,

$$\mathbf{D_k} = \begin{pmatrix} D_0 & . & D_k & 0 & 0 \\ 0 & 0 & . & . & 0 \\ . & . & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & & 0 \end{pmatrix}.$$

$$\mathbf{F_k} = \begin{pmatrix} F_0 & 0 & F_k & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & & 0 \end{pmatrix}$$, are N×N matrices of $d \times m$ matrices and the rest of the coefficients are defined by the following formulas

$\mathbf{K_k}(v_0, v_1, ..., v_{N-1})^T = (K_k v_0, ..., 0, ..., 0)^T \in (\mathrm{R}^m)^N$
for all $(v_0, v_1, ..., v_{N-1}) \in (\mathrm{R}^m)^N$ and
$\mathbf{C_k}(v_0, v_1, ..., v_{N-1})^T = (C_k v_0, 0, ..., 0)^T \in (\mathrm{R}^p)^N$
$\mathbf{S}(v_0, ..., v_{N-1})^T = (S v_0, 0, ..., 0)^T \in (\mathrm{R}^d)^N$
for all $(v_0, v_1, ..., v_{N-1}) \in (\mathrm{R}^d)^N$.

The real number $\varepsilon > 0$ is fixed and satisfies the condition

(2.2) $K_k > N\varepsilon$, for all $k = 0,.., N$,

$\mathbf{K_{mod}} = \mathrm{diag}(K_{N-1}, \frac{N-1}{N}K_{N-2} ..., \frac{2}{N}K_1, \frac{1}{N}K_0)$,

$\mathsf{K}_k^{\mathrm{mod},\varepsilon} = \mathrm{diag}(\frac{1}{N}K_k - \mathbf{I_N}\varepsilon, .., \frac{1}{N}K_0 - \mathbf{I_N}\varepsilon, ...,$
$\frac{\varepsilon}{N}\mathbf{I_N}, .., \frac{\varepsilon}{N}\mathbf{I_N})$,

The above Riccati equation is associated in [7] with the quadratic optimal control problem $\mathbf{O}_\varepsilon$ which consists in minimizing the cost functional

$$I_{x_0, N, \varepsilon}(U) = \sum_{k=0}^{N-2}(E\left[\left\langle \mathbf{C_k}^*\mathbf{C_k} X_k, X_k \right\rangle\right] + E\left[\left\langle \mathsf{K}_k^\varepsilon U_k, U_k \right\rangle\right]) + \left\langle \mathsf{K}_{N-1} U_{N-1}, U_{N-1} \right\rangle + E[\left\langle \left(\mathbf{C_{N-1}}^*\mathbf{C_{N-1}} + \mathbf{A}^*\mathbf{SA} + \mathbf{B}^*\mathbf{SB}\right) X_{N-1}, X_{N-1} \right\rangle + 2\left\langle \left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right) X_{N-1}, U_{N-1} \right\rangle$$

subject to a Grünwald-Letnikov fractional order system [3-4] of the form

$$(2.3) \qquad \Delta^{[\alpha]} x_{k+1} = \mathsf{A} x_k + \sum_{i=0}^{k} \xi_i \mathsf{B}_i x_i + \sum_{i=0}^{k} \mathsf{D}_i u_{k-i} + \sum_{i=0}^{k} \xi_i \mathsf{F}_i u_{k-i}, k \in \mathrm{N}$$
$$x_0 = x \in \mathrm{R}^d.$$

As proved in [7], Riccati equation (2.1) has a unique solution $\{R_n^\varepsilon\}_{n=0,..,N-1}$ and

$\min_{U \in \overline{\mathsf{U}}_{0,N-1}^a} I_{x_0, N, \varepsilon}(U) = E\left[\left\langle R_0^\varepsilon X_0, X_0 \right\rangle\right]$,
$X_0 = (x_0, 0, ..., \underset{N}{0})$, where $\overline{\mathsf{U}}_{0,N-1}^a$ is a modified class of admissible optimal controls (see[7]).

We also know from [7] that for any $a \le b$,

$$0 \le \left\langle R_0^a X_0, X_0 \right\rangle \le \left\langle R_0^b X_0, X_0 \right\rangle.$$

It follows that that limits $\lim_{b \to 0, b>0} \left\langle R_0^b X_0, X_0 \right\rangle$ and $\lim_{a \to 0, a<0} \left\langle R_0^a X_0, X_0 \right\rangle$ exists and are finite.

3. MAIN RESULTS

The goal of this section is to obtain some lower bounds of the function $(x_0, N) \to \inf_{U \in \mathsf{U}_{0,N-1}^a} I_{x_0, N}(U)$ for any $N > 0$ and $x_0 = x \in \mathrm{R}^d$.

Here

$$I_{x_0, N}(U) = E[\sum_{k=0}^{N-1}\left\langle \mathbf{C_k}^*\mathbf{C_k} X_k, X_k \right\rangle + \left\langle \mathbf{K} U_{N-1}, U_{N-1} \right\rangle + \left\langle \mathbf{S} X_N, X_N \right\rangle] \; X_0 = (x_0, 0, ..., \underset{N}{0})$$

is a cost functional to be minimized, associated to the linear fractional order system (2.3).

From Theorem 2 in [7], we know that $\inf_{u \in \mathsf{U}^a} I_{x_0, N}(u) \ge \lim_{\substack{b \to 0 \\ b<0}} \left\langle R_0^b X_0, X_0 \right\rangle$ for all N>0 and $X_0 = (x_0, 0, ..., \underset{N}{0}), x_0 = x \in \mathrm{R}^d$.

An approximating value of $\lim_{\varepsilon \to 0, \varepsilon<0} \left\langle R_0^\varepsilon X_0, X_0 \right\rangle$ can be obtained by computing $\left\langle R_0^\varepsilon X_0, X_0 \right\rangle$ for a increasing series of values of ε and by applying Lemma 2 from [7].

In order to do this, we write a MATLAB program for computing R_n^ε recursively, as indicated in (2.1).

The above MATLAB functions have general definitions, excepting the fact that (for the sake of simplicity) the values of some of the coefficients of the fractional order optimal control problem are assumed to be of a particular form (e.g B_k, D_k, F_k, C_k, K_k etc). Of course, these values could be read for a data file and the definition of the above functions will suffer minor changes.

Thus, let us introduce the following MATLAB functions, which define

a) the coefficient $c_i := (-1)^i \binom{\alpha}{i+1}$

***function** [r] = **cj**(alf,j)*
if j>1
*r = -cj(alf,j - 1) * (alf-j)/(j+1);*

```
else
r = -alf*(alf-1)/2;
end
end
```

and b) the coefficients $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D_k}$, $\mathbf{C_k}$, $\mathbf{F_k}$, $\mathbf{K_k}$, $\mathbf{S}$, $\mathbf{K_{mod}}$ and $K_k^{\mathrm{mod},\varepsilon}$ of the Riccati equation (2.1) for particular values of the coefficients of system (2.3).

```
function [rez] = matrA(h, alf,A,d,N)
A0=h^alf*A;
I=eye(d);
Z= zeros(d);
B0=I;
for i=1:N-1
   A0=[A0 I*cj( alf,i )];
   B0=[B0 Z];
end
A0=[A0;B0];
for i=3:N
   B0=[Z B0];
   B0=B0(:,1:d*N)
   A0=[A0;B0]
end
rez=A0;
end

function [rez] = matrB(d,N)

I=eye(d);
Z= zeros(d*(N-1),d*N);
A0=I;
for i=1:N-1
   A0=[A0 1/(i+1)*I];
end
rez=[A0;Z];
end

function [rez] = matrC(d,N,p)
C=(p+1)*(p+1)*[ 1 2];
for i=1:N-1
   C=[C zeros(1,2)];
end
Z= zeros(N-1,2*N);
rez=[C;Z];
end

function [rez] = matrD(d,N,K)
I=eye(d);
Z= zeros(d*(N-1),d*N);
A0=I;
if K~=0
for i=1:K
   A0=[A0 i/(i+1)*I];
end
for i=K+1:N-1
   A0=[A0 zeros(d)];
end
else
   for i=1:N-1
   A0=[A0 zeros(d)];
end;
end
rez=[A0;Z];
end

function [rez] = matrF(d,N,K)
I=eye(d);
Z= zeros(d*(N-1),d*N);
A0=I;
if K~=0
for i=1:K
   A0=[A0 2*i/(i+1)*I];
end
for i=K+1:N-1
   A0=[A0 zeros(d)];
end
else
   for i=1:N-1
   A0=[A0 zeros(d)];
end;
end
rez=[A0;Z];
end

function [rez] = matrK(d,N,p)
C=[ 2 1;1 2];
for i=1:N-1
   C=[C zeros(2)]
end
Z= zeros(2*(N-1),2*N);
C=[C;Z];
for i=2:N
   C(d*(i-1)+1:d*i,  d*(i-1)+1:d*i  )=  i*[  2
1;1 2];
end
rez=C;
end

function [rez] = matrS(d,N)
I=eye(d);
Z= zeros(d*(N-1),d*N);
A0=2*I;
```

```
p=zeros(d);
for i=1:N-1
   A0=[A0 p];
end
rez=[A0;Z];
end

function [rez] = matrKmod(d,N,p)
C=[ 2 1;1 2];
for i=1:N-1
   C=[C zeros(2)]
end
Z= zeros(2*(N-1),2*N);
C=[C;Z];
for i=1:N-1
   C(d*(i)+1:d*(i+1), d*(i)+1:d*(i+1) )=((N-i)/N)*(i+1)*[ 2 1;1 2];
end
rez=C;
end

function [rez] = matrKmodK1(d,N,p,eps)
%K=1/N*[ 2 1;1 2];
Z= -eps*eye(2*N,2*N);
for i=0:p
Z(2*i+1:2*(i+1),2*i+1:2*(i+1))=1/N*(i+1)*[ 2 1;1 2] ;
end
rez=Z;
end
```

Function **Opt**(eps,N) returns the value $\left\langle R_0^{\varepsilon} X_0, X_0 \right\rangle$ for a given value of the variables eps and N.

```
function[rez]=Opt(eps,N)
alf=1/2;
A=[1 2; 0 1];
d=2;
h=2;
p=1;
x0=[1 ;2];

A00=matrA(h, alf,A,d,N);
B=matrB(d,N);
S=matrS(d,N);
R(:,:,N)=matrC(d,N,N-1)'*matrC(d,N,N-1)
+A00'*S*A00+B'*S*B-(matrD(d,N,N-1)'
*S*A00+matrF(d,N,N-1)'
*S*B)'*inv(matrKmod(d,N,N)+matrD(d,N,N-1)'*S*matrD(d,N,N-1)+matrF(d,N,N-1)'
*S*matrF(d,N,N-1))*(matrD(d,N,N-1)'
*S*A00+matrF(d,N,N-1)'*S*B)
S=R(:,:,N);
for k=N-2:-1:0
R(:,:,k+1)=matrC(d,N,k)'*matrC(d,N,k)+A00
'*S*A00+B'*S*B-
(matrD(d,N,k)'*S*A00+matrF(d,N,k)'*S*B)'*
inv(matrKmodK1(d,N,k,eps)+matrD(d,N,k)'*
S*matrD(d,N,k)+matrF(d,N,k)'*S*matrF(d,N,
k))*(matrD(d,N,k)'*S*A00+matrF(d,N,k)'*S*
B)
S=R(:,:,k+1);
end
copt=x0'*R(1:2,1:2,1)*x0;
rez=copt
```

For N = 10 and eps= 0.0035:-0.001.0.001 we get the following representations of functions $\varepsilon \xrightarrow{f} \left\langle R_0^{\varepsilon} X_0, X_0 \right\rangle$ and $\varepsilon \xrightarrow{g} \left\langle R_0^{-\varepsilon} X_0, X_0 \right\rangle$.

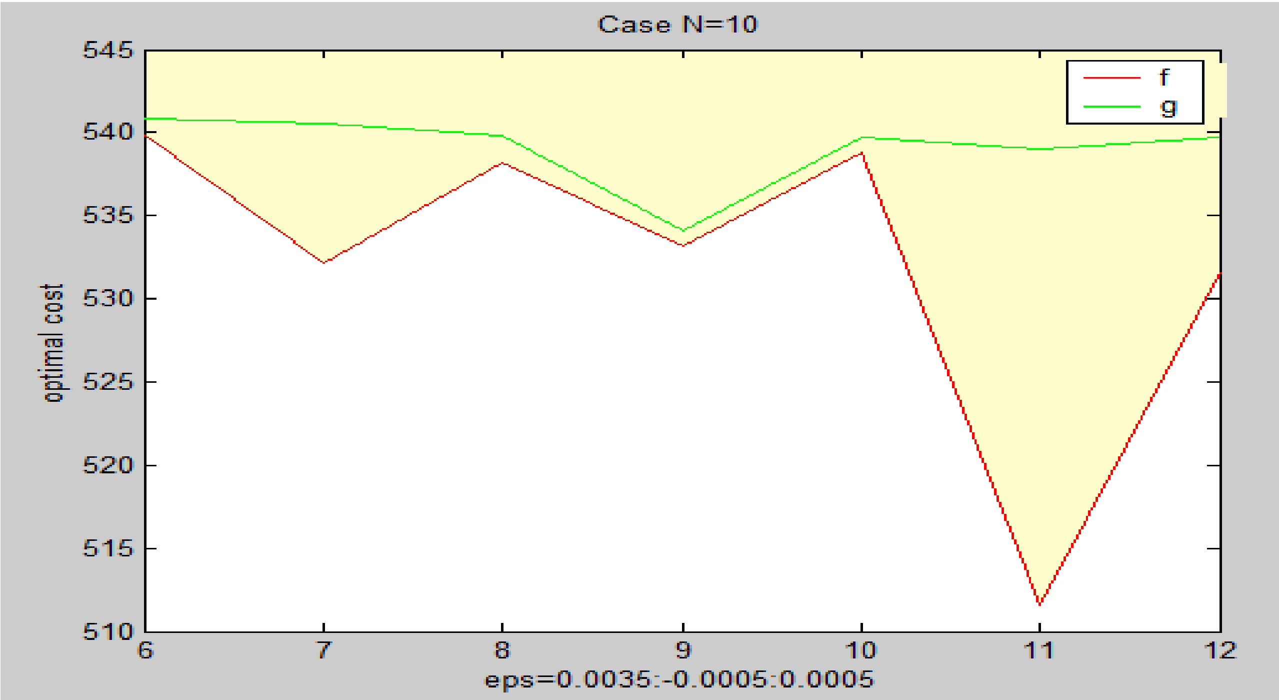

Figure 1. The graph of the two functions f and g for N=10,

In view of Remark 1 from [7], we know that f(ε) ≤ g(ε), as we can see in the above picture. The red line in Fig. 1 represent a lower bound of the optimal value of the cost $I_{x_0,N}(u)$, if it exists. We also note that $\langle R_0^{\varepsilon} X_0, X_0 \rangle$ is well defined for any ε satisfying condition (2.2). In our case, K_k =1/(k+1) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, k∈{0,1..,10}, and (2.2) which implies that ε<1/110 ≈ 0.009.

Reasoning as in Remark 1 from [7], and replacing $\mathsf{K}_k^{\mathrm{mod},\varepsilon}$ by $\mathrm{diag}(\frac{1}{N}K_k,..., \frac{1}{N}K_0,...., -\varepsilon\mathbf{I_N},..,-\varepsilon\mathbf{I_N})$ it follows that

$$I_{x_0,N,\varepsilon}(U) = \sum_{k=0}^{N-2}(E\left[\left\langle \mathbf{C_k}^*\mathbf{C_k} X_k, X_k \right\rangle\right] + E\left[\left\langle -\varepsilon \mathbf{I_N} U_k, U_k \right\rangle\right]) + E[\left\langle \left(\mathbf{C_{N-1}}^*\mathbf{C_{N-1}} + \mathbf{A}^*\mathbf{SA} + \mathbf{B}^*\mathbf{SB}\right) X_{N-1}, X_{N-1} \right\rangle + 2\left\langle \left(\mathbf{D}_{N-1}^*\mathbf{SA} + \mathbf{F}_{N-1}^*\mathbf{SB}\right) X_{N-1}, U_{N-1} \right\rangle + \left\langle \mathsf{K}_{N-1} U_{N-1}, U_{N-1} \right\rangle]$$

is another lower bound of the cost $I_{x_0,N,\varepsilon}(U)$.

Let us consider the function $\varepsilon \xrightarrow{h} \langle R_0^{\varepsilon} X_0, X_0 \rangle$, where R_0^{ε} is the first component of the solution of the Riccati equation defined by the new value of $\mathsf{K}_k^{\mathrm{mod},\varepsilon}$ and the above cost function. It also represent a lower bound of the optimal value of the cost $I_{x_0,N}(u)$, if it exists.

The graph of the two functions f and h are depicted in Fig. 2. We note that the graph a function *max{f,g}* is a new lower bound of $I_{x_0,N}(u)$.

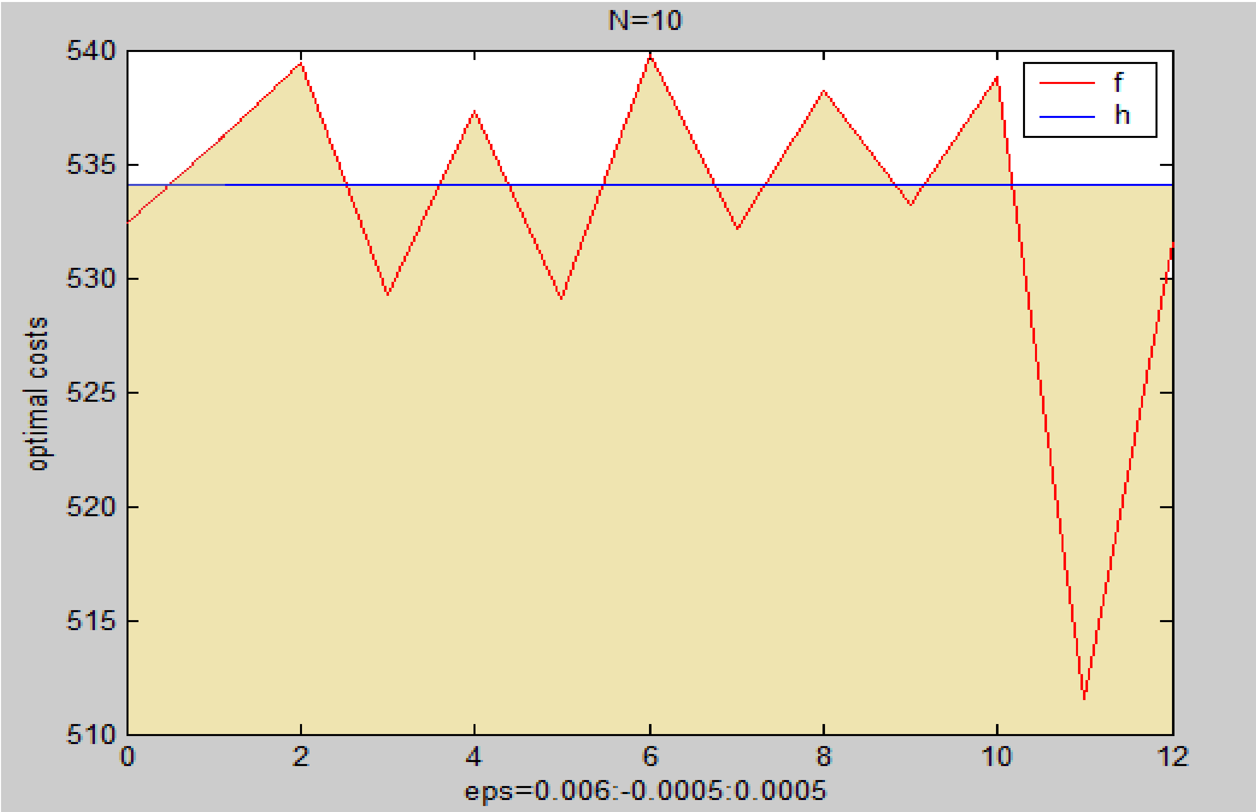

Figure 2. The graphs of the functions f and h for N=10

4. CONCLUSION

This paper presents two lower approximations of the cost functional $I_{x_0,N}(u)$. Each of them defines an optimal control problem which solution generates a lower bound for $I_{x_0,N}(u)$. We have used MATLAB programs to solve the associated Riccati equations of control and to generate the graphics of the lower bounds of the cost functional $I_{x_0,N}(u)$.

5. BIBLIOGRAPHY

[1] Anderson, B. D. A., Moore, J. B. *Optimal control: linear quadratic methods*, Prentice Hall, Englewood Cliffs, 1990.

[2] Dragan, V. ,Morozan, T., Stoica, A. *Mathematical Methods in Robust Control of Discrete Time Linear Stochastic* Systems, Springer, New York, 2013.

[3] Kilbas,, A.A, Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*. Elsevier Science, Amsterdam, 2006.

[4] Monje, C.A. , et al. *Fractional-order Systems and Controls: Fundamentals and Applications*, Springer Science & Business Media, London, 2010.

[5] Ungureanu, V. M. , *Stability, stabilizability and detectability for Markov jump discrete-time linear systems with multiplicative noise in Hilbert spaces*, Optimization 63.11 (2014): 1689-1712

[6] Ungureanu, V. M., Dragan, V. ,Morozan, T.,, *Global solutions of a class of discrete-time backward nonlinear equations on ordered Banach spaces with applications to Riccati equations of stochastic control*, Optimal Control Applications and Methods, 34(2013), 2, 164-

[7] V Ungureanu, V. M., *Approximating solutions for a class of stochastic fractional linear quadratic optimal control problems, Annals of' Constantin Brancusi University of Targu-Jiu. Engineeringe Series* 2 (2018) 6p(to appear).