FUZZY GROUPOID MORPHISMS AND ACTIONS

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ABSTRACT: *The purpose of this paper is to introduce a novel concept of groupoid action and to obtain a characterization of a groupoid homomorphism as an action that commutes with the multiplication on the range. We also define the corresponding fuzzy concepts. This approach allows the study within a unified framework of various fuzzy structures viewed as fuzzy subgroupoids of appropriate groupoids.*

KEY WORDS: *groupoid, homomorphism, groupoid action, fuzzy subgroupoid, fuzzy action, fuzzy homomorphism*

1. TERMINOLOGY, NOTATION AND PRELIMINARIES

In this paper by a groupoid we mean a small category with inverses (notion introduced by Brandt [2]). More precisely, by a (crisp) groupoid we mean a set Γ, together with a partially defined multiplication

$$(\gamma_1, \gamma_2) \to \gamma_1\gamma_2 \; [: \Gamma^{(2)} \subset \Gamma \times \Gamma \to \Gamma],$$

and an inverse map

$$x \to x^{-1} \; [: \Gamma \to \Gamma],$$

satisfying the following properties:

1. $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \Gamma$.
2. If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and $(\gamma_2, \gamma_3) \in \Gamma^{(2)}$, then $(\gamma_1\gamma_2, \gamma_3), (\gamma_1, \gamma_2\gamma_3) \in \Gamma^{(2)}$ and $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$.
3. For all $\gamma \in \Gamma$, $(\gamma, \gamma^{-1}), (\gamma^{-1}, \gamma) \in \Gamma^{(2)}$, and if $(\gamma, \gamma')\in G^{(2)}$ (respectively, $(\gamma'', \gamma)\in G^{(2)}$), then $\gamma' = x^{-1}(x\gamma')$ (respectively, $\gamma'' = (\gamma''\gamma)\gamma^{-1}$).

For every groupoid Γ, the maps r and d defined by $r(\gamma) =\gamma\gamma^{-1}$ and respectively, $d(\gamma) = \gamma^{-1}\gamma$, are called the range and respectively the domain/source map. This maps have a common image $r(\Gamma) = d(\Gamma)$ called the unit space of Γ and denoted $\Gamma^{(0)}$. It is useful to note that $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ if and only if $d(\gamma_1) = r(\gamma_2)$. In terms of categories the objects or units of the groupoid (seen as a category) may be identified with $\Gamma^{(0)}$ and the morphisms (arrows) with Γ.

The concept of fuzzy set was introduced by Zadeh [14] and since then it have been used to model uncertain information in various areas, and the theory of fuzzy set was extended to other algebraic structure.

We propose in this paper a notion of generalized fuzzy action and fuzzy groupoid homomorphism based on the notion of fuzzy subgroupoid introduced in [5]. This approach allows the study within a unified framework of various fuzzy structures such as fuzzy sets [14], fuzzy subgroup [13, 8] equivalence relation [7-10], group actions [1], etc.

2. A NEW CONCEPT OF GROUPOID ACTION

Let us start this section recalling some classical constructions of groupoids. Let Γ be a groupoid and $f : X \to \Gamma^{(0)}$ be a map. Then

$$\Gamma[f, X] = \{(x, \gamma, y) \in X \times \Gamma \times X: f(x) = r(\gamma) \text{ and } f(y) = d(\gamma)\}$$

becomes a groupoid under the operations:

$$(x, \gamma_1, y)(y, \gamma_2, z) = (x, \gamma_1\gamma_2, y)$$
$$(x, \gamma, y)^{-1} = (x, \gamma^{-1}, y).$$

($\Gamma[f, X]$ is obtained by blowing up the unit space of Γ to $f(X)$).

A classical left action of a groupoid Γ on a set X is given by two maps: a map $\rho : X \to \Gamma^{(0)}$ (called a momentum map) and a map

$$(\gamma, x) \mapsto \gamma \cdot x$$

from

$$\{(\gamma, x) \in \Gamma \times X: d(\gamma) = \rho(x)\}$$

to X, called left action, satisfying the following condition:

1. $\rho(\gamma \cdot x) = r(\gamma)$ for all $\gamma \in \Gamma$ and $x \in X$ such that $d(\gamma) = \rho(x)$.
2. $\rho(x) \cdot x = x$ for all $x \in X$.
3. If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, $x \in X$ and $d(\gamma_2) = \rho(x)$, then $(\gamma_1\gamma_2) \cdot x = \gamma_1 \cdot (\gamma_2 \cdot x)$.

In the same manner, we define a right action of Γ on X, using a map $\sigma : X \to \Gamma^{(0)}$ and a map

$$(x, \gamma) \mapsto x \cdot \gamma$$

from

$$\{(x, \gamma) \in X \times \Gamma : \sigma(x) = r(\gamma)\}$$

to X satisfying the following conditions:

1. $\sigma(x \cdot \gamma) = d(\gamma)$ for all $\gamma \in \Gamma$ and $x \in X$ such that $r(\gamma) = \sigma(x)$.
2. $x \cdot \sigma(x) = x$ for all $x \in X$.
3. If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, $x \in X$ and $r(\gamma_1) = \sigma(x)$, then $x \cdot (\gamma_1\gamma_2) = (x \cdot \gamma_1) \cdot \gamma_2$.

If $(\gamma, x) \mapsto \gamma \cdot x$ is a classical left action of a groupoid Γ on a set X with the momentum map ρ, then

$$\Gamma \ltimes X = \{(\gamma \cdot x, \gamma, x) \in X \times \Gamma \times X: r(\gamma) = \rho(x), d(\gamma) = \rho(y)\}$$

can be viewed as a subgroupoid of $\Gamma[X, \rho]$.

Similarly, if $(x, \gamma) \mapsto x \cdot \gamma$ is a classical right action of a groupoid Γ on a set X with the momentum map σ, then

$$X \rtimes \Gamma = \{(x, \gamma, x \cdot \gamma) \in X \times \Gamma \times X: r(\gamma) = \sigma(x), d(\gamma) = \sigma(x)\}$$

is a subgroupoid of $\Gamma[X, \sigma]$.

If Γ and G are two groups and $f: \Gamma \to G$ is a group homomorphism, then

$$\gamma \cdot x = f(\gamma)x$$

defines an action of Γ on G (in the groupoid terms the momentum map is given by $x \mapsto e$, where e is identity element of Γ). In the case of a groupoid homomorphism, $f: \Gamma \to G$, the formulae $\gamma \cdot x = f(\gamma)x$ does not necessarily define an action, due to the impossibility of defining the momentum map. We propose a new notion of groupoid action to be able to overcome this impediment.

We say a groupoid Γ acts to the left on a set X if there is a set S and three maps: $\rho : X \to S$, $\rho^{(0)} : \Gamma^{(0)} \to S$ and

$$(\gamma, x) \mapsto \gamma \cdot x$$

from

$$\{(\gamma, x) \in \Gamma \times X: \rho^{(0)}(d(\gamma)) = \rho(x)\}$$

to X, called left action, satisfying the following condition:

1. $\rho(\gamma \cdot x) = \rho^{(0)}(r(\gamma))$ for all $\gamma \in \Gamma$ and $x \in X$ such that $\rho^{(0)}(d(\gamma)) = \rho(x)$.
2. $u \cdot x = x$ for all $x \in X$ and all $u \in \Gamma^{(0)}$ such that $\rho^{(0)}(u) = \rho(x)$.
3. If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, $x \in X$ and $\rho^{(0)}(d(\gamma_2)) = \rho(x)$, then $(\gamma_1\gamma_2) \cdot x = \gamma_1 \cdot (\gamma_2 \cdot x)$.

A classical left action of Γ on X corresponds to the particular case when S is $\Gamma^{(0)}$ and $\rho^{(0)}$ is the identity function. In particular the left multiplication on Γ induces an action

$$\gamma \cdot x = \gamma x$$

with $S = \Gamma^{(0)}$, $\rho = r$, and $\rho^{(0)}$ the identity function on $\Gamma^{(0)}$.

If $(\gamma, x) \mapsto \gamma \cdot x$ is left action of a groupoid Γ on a set X in the sense introduce in this paper, then

$$\Gamma \ltimes_* X = \{((\gamma \cdot x, r(\gamma)), \gamma, (x, d(\gamma))) \in X \times \Gamma \times X: \rho^{(0)}(d(\gamma)) = \rho(x)\}$$

can be viewed as a subgroupoid of

$$\Gamma[X * \Gamma^{(0)}, pr_2],$$

where

$$X * \Gamma^{(0)} = \{(x, u): \rho^{(0)}(u) = \rho(x)\}$$

and $pr_2(x, u) = u$ for all $(x, u) \in X * \Gamma^{(0)}$.

In the same manner, we define a right action of Γ on X, using a set S and three maps: $\sigma : X \to \Gamma^{(0)}$, $\sigma^{(0)} : \Gamma^{(0)} \to S$ and

$$(x, \gamma) \mapsto x \cdot \gamma$$

from

$$\{(x, \gamma) \in X \times \Gamma : \sigma(x) = r(\gamma)\}$$

to X. called right action, satisfying the following conditions:

1. $\sigma(x \cdot \gamma) = \sigma^{(0)}(d(\gamma))$ for all $\gamma \in \Gamma$ and $x \in X$ such that $\sigma^{(0)}(r(\gamma)) = \sigma(x)$.
2. $x \cdot u = x$ for all $x \in X$ and all $u \in \Gamma^{(0)}$ such that $\sigma^{(0)}(u) = \sigma(x)$.
3. If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, $x \in X$ and $\sigma^{(0)}(r(\gamma_1)) = \sigma(x)$, then $x \cdot (\gamma_1\gamma_2) = (x \cdot \gamma_1) \cdot \gamma_2$.

As in the left action case, a classical right action of Γ on X corresponds to the particular case when S is $\Gamma^{(0)}$ and $\sigma^{(0)}$ is the identity function. In particular the right multiplication on Γ induces an action

$$x \cdot \gamma = x\gamma$$

with $S = \Gamma^{(0)}$, $\sigma = d$, and $\sigma^{(0)}$ the identity function on $\Gamma^{(0)}$.

Also if $(x, \gamma) \mapsto x \cdot \gamma$ is right action of a groupoid Γ on a set X in the sense introduce in this paper, then

$$X \ast\!\rtimes \Gamma = \{((x, r(\gamma)), \gamma, (x\cdot\gamma, d(\gamma))) \in X\times\Gamma\times X: \sigma^{(0)}(r(\gamma)) = \sigma(x)\}$$

can be viewed as a subgroupoid of

$$\Gamma[X*\Gamma^{(0)}, pr_2],$$

where

$$X*\Gamma^{(0)} = \{(x, u): \sigma^{(0)}(u) = \sigma(x)\}$$

and $pr_2(x, u) = u$ for all $(x, u) \in X*\Gamma^{(0)}$.

Let $(\gamma, x) \mapsto \gamma \cdot x$ be a left action of a groupoid Γ on a set X and let $(x, \zeta) \mapsto x \cdot \zeta$ be a right action of a groupoid G on X and in the sense introduce in this paper. We say that the two actions commutes if the following conditions are satisfied:

1. $\rho(x \cdot \zeta) = \rho(x)$ for all $\zeta \in G$ and $x \in X$ such that $\sigma(x) = \sigma^{(0)}(r(\zeta))$.
2. $\sigma(\gamma \cdot x) = \sigma(x)$ for all $\gamma \in \Gamma$ and $x \in X$ such that $\rho(x) = \rho^{(0)}(d(\gamma))$.
3. $\gamma \cdot(x \cdot \zeta) = (\gamma \cdot x) \cdot \zeta$ for all $\gamma \in \Gamma$, $\zeta \in G$ and $x \in X$ such that $\rho(x) = \rho^{(0)}(d(\gamma))$ and $\sigma(x) = \sigma^{(0)}(r(\zeta))$.

3. GROUPOID HOMOMORPHISMS AS ACTIONS

Let Γ and G be groupoids. A function

$$f : \Gamma \to G$$

is a (groupoid) homomorphism if $(f(\gamma_1), f(\gamma_2)) \in G^{(2)}$ and

$$f(\gamma_1)f(\gamma_2) = f(\gamma_1\gamma_2)$$

whenever $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$.

Let us remark that since

$$f(\gamma_1^{-1})f(\gamma_1)f(\gamma_2) = f(\gamma_1^{-1}\gamma_1\gamma_2) = f(\gamma_2),$$

it follows that $f(\gamma_1^{-1}) = f(\gamma_1)^{-1}$. Hence

$$f(\gamma_1\gamma_1^{-1}) = f(\gamma_1)f(\gamma_2)^{-1} \in \Gamma^{(0)},$$

and consequently, $f(\Gamma^{(0)}) \subset G^{(0)}$.

If $f: \Gamma \to G$ is a groupoid homomorphism, then

$$\gamma \cdot x = f(\gamma)x$$

defines a left action of Γ on G if we take $S = G^{(0)}$, $\rho = r$, $\rho^{(0)} = f|\Gamma^{(0)}$.

For all $x, y \in G$ such that $d(x) = r(y)$, we have

$$r(xy) = r(x)$$

Also for all $\gamma \in \Gamma$ and $x \in G$ such that

$$r(x) = f(\gamma)) = f(d(\gamma)) = \rho^{(0)}(d(\gamma))$$

we have

$$d(\gamma \cdot x) = d(f(\gamma)x) = d(x)$$

Furthermore if $\gamma \in \Gamma$and $x, y \in G$ are such that $r(x) = \rho^{(0)}(d(\gamma))= f(d(\gamma))$ and $d(x) = r(y)$, then

$$(\gamma \cdot x)y = (f(\gamma)x)y = f(\gamma)(xy) = \gamma \cdot (xy)/$$

Consequently,

$$(\gamma, x) \mapsto \gamma \cdot x = f(\gamma)x$$

defines a left action of Γ on G that commutes with the right multiplication on G.

Similarly,

$$x \cdot \gamma = xf(\gamma)$$

defines a right action of Γ on G if we take $S = G^{(0)}$, $\sigma = d$, $\sigma^{(0)} = f|\Gamma^{(0)}$. This action commutes with the left multiplication on G.

Conversely, let us assume that we have a left action

$$(\gamma, x) \mapsto \gamma \cdot x$$

of a groupoid Γ on a groupoid G in the sense introduce in this paper such that:

1. $\rho = r$ the range map of G
2. $(\gamma, x) \mapsto \gamma \cdot x$ commutes with the multiplication on G.

In this case let us define

$$f(\gamma) = \gamma \cdot \rho^{(0)}(d(\gamma))$$

for all $\gamma \in \Gamma$. If $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, then

$$r(\gamma_2 \cdot \rho^{(0)}(d(\gamma_2)) = \rho(\gamma_2 \cdot \rho^{(0)}(d(\gamma_2)) = \rho^{(0)}(r(\gamma_2)) = \rho^{(0)}(d(\gamma_1))$$

and

$$d(\gamma_1 \cdot \rho^{(0)}(d(\gamma_1)) = \sigma(\gamma_1 \cdot \rho^{(0)}(d(\gamma_1)) = \sigma(\rho^{(0)}(d(\gamma_1))) = \rho^{(0)}(d(\gamma_1))$$

(we took into account the fact $\rho = r$ the range map of G and $\sigma = d$ the domain/source map of G). Consequently, if $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, then $(f(\gamma_1), f(\gamma_2)) \in G^{(2)}$.

Since the left action $(\gamma, x) \mapsto \gamma \cdot x$ commutes with the right multiplication on G, it follows that

$$(\gamma_1 \cdot \rho^{(0)}(d(\gamma_1)))(\gamma_2 \cdot \rho^{(0)}(d(\gamma_2))) = \gamma_1 \cdot (\rho^{(0)}(d(\gamma_1))(\gamma_2 \cdot \rho^{(0)}(d(\gamma_2)))) = \gamma_1 \cdot (\gamma_2 \cdot \rho^{(0)}(d(\gamma_2)))$$

Thus we have

$$f(\gamma_1)f(\gamma_2) = (\gamma_1 \cdot \rho^{(0)}(d(\gamma_1)))(\gamma_2 \cdot \rho^{(0)}(d(\gamma_2))) = \gamma_1 \cdot (\gamma_2 \cdot \rho^{(0)}(d(\gamma_2))) = (\gamma_1\gamma_2) \cdot \rho^{(0)}(d(\gamma_1\gamma_2))) = f(\gamma_1\gamma_2)$$

Consequently, f is a groupoid homomorphism. Moreover the action associated to this homomorphism coincides with the initial action:

$$f(\gamma)x = (\gamma \cdot \rho^{(0)}(d(\gamma)))x = \gamma \cdot(\rho^{(0)}(d(\gamma))x) = \gamma \cdot x$$

(we took into consideration that the action commutes with the right multiplication on G) Also the homomorphism associated to the left action induced by a groupoid homomorphism $f : \Gamma \to G$ coincides with f. Indeed,

$$\gamma \cdot \rho^{(0)}(d(\gamma)) = f(\gamma)\rho^{(0)}(d(\gamma)) = f(\gamma)$$

for all $\gamma \in\Gamma$.

Therefore any groupoid homomorphism

$$f : \Gamma \to G$$

can be viewed as a left action $(\gamma, x) \mapsto \gamma \cdot x$ of Γ on G (in the sense introduced in this paper) which have the following properties:

i. $\rho = r$ = the range map of G
ii. $(\gamma, x) \mapsto \gamma \cdot x$ commutes with the multiplication on G.

An action in classical sense of Γ on G which commutes with the multiplication on G is used in [4, 3] to define a notion of morphism/arrow from Γ to G that allows the constructions of categories whose objects are groupoids. However the two notions are different. For the action used in [3, 4]

$$\rho : G \to \Gamma^{(0)}$$

is arbitrary and for the action used here we have two maps $\rho = r : G \to G^{(0)}$ and $\rho^{(0)} : \Gamma^{(0)} \to G^{(0)}$ in order to establish the pairs (γ, x) such that γ can act on x.

4. FUZZY ACTIONS AND HOMOMORPHISMS

Let I = [0,1] (or more generally, a bounded lattice) and $T: I \times I \to I$ be a function. In particular, T can be a t-norm, i.e. a function

$$T: [0, 1] \times [0, 1] \to [0, 1]$$

which satisfies the following properties:

1. $T(a, b) = T(b, a)$ for all $a,b\in[0,1]$;
2. $T(a, b) \le T(c, d)$ if $a \le c$ and $b \le d$;
3. $T(a, T(b, c)) = T(T(a, b), c)$ for all $a,b,c\in[0,1]$;
4. $T(a, 1) = a$ for all $a\in[01,]$

Let us note that if T is a t-norm, then $T \le T_{min}$, where $T_{min}(a,b) = \min\{a,b\}$ for all $a,b\in[0,1]$ (see [11, 12] for more examples of t-norms and applications). Let us recall the definition introduces in [5] for a fuzzy subgroupoid of a groupoid Γ:

A function $\mu: \Gamma \to I$ is said to be T-fuzzy subgroupoid of Γ if the following conditions are satisfied

1. $\mu(\gamma_1\gamma_2) \ge T(\mu(\gamma_1), \mu(\gamma_2))$ for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$.
2. $\mu(\gamma^{-1}) \ge \mu(\gamma)$ for all $\gamma\in\Gamma$.
3. $\mu(r(\gamma)) \ge \mu(\gamma)$ for all $\gamma \in \Gamma$.

If $T(a,b) = T_{min}(a,b) = \min\{a, b\}$ for all a, b I, then condition 3 in the definition of fuzzy groupoid is automatically satisfied.

In [6] we define a fuzzy intuitionistic left (respectively, right) action of a groupoid Γ on a set X as a fuzzy intuitionistic subgroupoid of $\Gamma \ltimes X$ (respectively, $X \rtimes \Gamma$). Using the same ideas we define a fuzzy left action of a groupoid Γ as fuzzy subgroupoid of

$$\Gamma[X{*}\Gamma^{(0)}, pr_2],$$

where

$$X{*}\Gamma^{(0)} = \{(x, u): \rho^{(0)}(u) = \rho(x)\}$$

$\rho : X \to S$, $\rho^{(0)} : \Gamma^{(0)} \to S$, S is a set and

$$pr_2(x, u) = u \text{ for all } (x, u) \in X{*}\Gamma^{(0)}.$$

In the subsequent we identify $\Gamma[X{*}\Gamma^{(0)}, pr_2]$ with

$$\Gamma[X, \rho, \rho^{(0)}] =\{(x, \gamma, y) \in X \times \Gamma \times X: \rho^{(0)}(r(\gamma)) = \rho(x), \quad \rho^{(0)}(d(\gamma)) = \rho(y)\}$$

by $(x, r(\gamma), \gamma, y, d(\gamma) \mapsto (x, \gamma, y)$.

Identifying $\Gamma[X{*}\Gamma^{(0)}, pr_2]$ with

$$\Gamma[X, \rho, \rho^{(0)}] =\{(x, \gamma, y) \in X \times \Gamma \times X: \rho^{(0)}(r(\gamma)) = \rho(x), \quad \rho^{(0)}(d(\gamma)) = \rho(y)\},$$

a T-fuzzy left action of a groupoid Γ on a set X is a function

$\mu: \Gamma[X, \rho, \rho^{(0)}] \to I$

satisfying the following conditions

1. $\mu(x, \gamma_1\gamma_2, z) \geq T(\mu(x, \gamma_1, y), \mu(y, \gamma_2, z))$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $x, y, z \in X$ such that $d(\gamma_1) = r(\gamma_2)$, $\rho^{(0)}(r(\gamma_1)) = \rho(x)$, $\rho^{(0)}(d(\gamma_1)) = \rho(y)$, $\rho^{(0)}(r(\gamma_2)) = \rho(y)$, $\rho^{(0)}(d(\gamma_2)) = \rho(z)$.
2. $\mu(y, \gamma^{-1}, x) \geq \mu(x, \gamma, y)$ for all $\gamma \in \Gamma$ and $x, y \in X$ such that $\rho^{(0)}(r(\gamma)) = \rho(x)$, $\rho^{(0)}(d(\gamma)) = \rho(y)$.
3. $\mu(x, r(\gamma), x)) \geq \mu(x, \gamma, y)$ for all $\gamma \in \Gamma$ and $x, y \in X$ such that $\rho^{(0)}(r(\gamma)) = \rho(x)$, $\rho^{(0)}(d(\gamma)) = \rho(y)$.

If $T(a,b) = T_{min}(a,b) = \min\{a, b\}$ for all $a, b \in I$, then condition 3 in the above definition of fuzzy action is automatically satisfied.

In the above definition

$\mu(x, \gamma, y)$

can be viewed as the degree to which x is the result of the action of γ on y (i.e. $\gamma \cdot y$).

In the particular case when

i. Γ is a group
ii. $\rho : X \to \{e\}$, $\rho^{(0)} : \{e\} \to \{e\}$
iii. $\mu(x, e, x) = 1$ for all $x \in X$,

if we define

$\alpha(\gamma, x, y) = \mu(y, \gamma, x)$

for all $x, y \in X$ and $\gamma \in \Gamma$, then $\alpha: \Gamma \times X \times X \to I$ is a T –fuzzy action in the sense of [1].

Similarly, we define a fuzzy right action of a groupoid Γ as fuzzy subgroupoid of

$\Gamma[X*\Gamma^{(0)}, pr_2]$,

where

$X*\Gamma^{(0)} = \{(x, u): \sigma^{(0)}(u) = \sigma(x)\}$

$\sigma : X \to S$, $\sigma^{(0)} : \Gamma^{(0)} \to S$, S is a set and

$pr_2(x, u) = u$ for all $(x, u) \in X*\Gamma^{(0)}$.

We have proved in the preceding section that a groupoid homomorphism $f : \Gamma \to G$ can be seen as a left action (in the sense introduced in this paper) of Γ on G which commutes with the right multiplication on G.

We propose the following definition for a fuzzy homomorphism of groupoids: a fuzzy action of Γ on G,

$\mu: \Gamma[G, r, \rho^{(0)}] \to I$

that commutes with a fuzzy action of G on itself by right multiplication.

A fuzzy action of G on itself by right multiplication is a function

$\mu_G : G[G, d, id] \to I$,

where

$G[G, d, id] = \{(x, y, z) \in G \times G \times G: r(y) = d(x) \text{ and } d(y) = r(z)\}$

satisfying the following conditions:

1. $\mu_G(x, y_1y_2, z) \geq T(\mu(x, y_1, y), \mu(y, y_2, z))$ for all $y_1, y_2, x, y, z \in G$ such that $d(y_1) = r(y_2)$, $r(y_1) = d(x)$, $d(\gamma_1) = r(y)$, $r(\gamma_2) = d(y)$, $d(\gamma_2) = r(z)$.
2. $\mu_G(z, y^{-1}, x) \geq \mu_G(x, y, z)$ for all $x, y, z \in G$ such that $d(x) = r(y)$ and $r(y) = d(z)$.
3. $\mu_G(x, r(y), x)) \geq \mu_G(x, y, z)$ for all $x, y, z \in G$ such that $d(x) = r(y)$ and $r(y) = d(z)$.

The fuzzy action of Γ on G

$\mu: \Gamma[G, r, \rho^{(0)}] \to I$

commutes with

$\mu_G : G[G, d, id] \to I$,

if and only if

1. $\mu(t, \gamma, w) \geq T(\mu(z, \gamma, x), \mu_G(z, y, t), \mu_G(x, y, w))$ for all $\gamma \in \Gamma$ and $x, y, z, t, w \in G$ such that $d(x) = r(y)$, $d(y) = r(w)$, $d(z) = r(y)$, $\rho^{(0)}(d(\gamma)) = r(x) = r(w)$, $\rho^{(0)}(r(\gamma)) = r(z) = r(t) = d(y)$.
2. $\mu_G(w, y, t) \geq T(\mu_G(x, y, z), \mu(t, \gamma, z), \mu(w, \gamma, x))$ for all $\gamma \in \Gamma$ and $x, y, z, t, w \in G$ such that $d(x) = r(y)$, $d(y) = r(z) = r(x) = \rho^{(0)}(d(\gamma))$, $\rho^{(0)}(r(\gamma)) = r(t) = r(w)$,

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