A NOTE ON NONAUTONOMOUS DISCRETE SYSTEMS IN TERMS OF GROUPOIDS

Mădălina Roxana Buneci, Associate Professor

"Constantin Brâncuși" University from Târgu Jiu, ROMANIA

ABSTRACT: *The purpose of this paper is to associate a family of groupoids and morphisms to a discrete nonautonomous system that does not necessarily satisfy the semigroup property. This approach provides a unifying framework to study discrete nonautonomous systems and to uncover similarities between seemingly unrelated systems.*
KEY WORDS: *discrete nonautonomous system, groupoid, homomorphism*

1. TERMINOLOGY, NOTATION AND PRELIMINARIES

The mathematical formalization [7] for a nonautonomous discrete-time process consists of a space X and a sequence $(f_n)_n$ of maps $f_n : X \to X$. Then the nonautonomous difference equation

$$x_{n+1} = f_n(x_n)$$

generates a discrete-time process

$$f : (\mathbf{N} \times_{\geq} \mathbf{N}) \times X \to X$$

by:

$$f(n_0, n_0, x) := x,$$

$$f(n, n_0, x) := f_{n-1} \circ f_{n-2} \circ ... \circ f_{n_0}(x).$$

where

$$\mathbf{N} \times_{\geq} \mathbf{N} = \{(n, n_0) \in \mathbf{N} \times \mathbf{N}: n \geq n_0\}.$$

The function f: $\mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$ defined above has the following properties:

1.1) $f(n_0, n_0, x) = x$ for all $n_0 \in \mathbf{N}$ and $x \in X$.

1.2) $f(n_2, n_0, x) = f(n_2, n_1, f(n_1, n_0, x))$ for all $n_0 \leq n_1 \leq n_2$ and $x \in X$.

The property 1.2 allows us to define an action of $\mathbf{N}$ on $\widetilde{X}$ by

$$n \cdot (n_0, x) := (n + n_0, f(n + n_0, n_0, x))$$

for all $n \in \mathbf{N}$ and $(n_0, x) \in \widetilde{X}$.

Thus if a process f satisfies semigroup property 1.2, then some of its properties can also be studied in the groupoid framework as in [5] or as an object in a groupoid category as in [1]. However, no compact set in X is invariant under this action.

Moreover there are processes (such as the fractional order systems [8]) that do not satisfy semigroup condition 1.2. In this article we consider processes that do not necessarily satisfy semigroup condition 1.2. More precisely, we take into consideration the processes generated by a difference equations of the form

$$x_{n+1} = f_{n,n_0}(x_n, x_{n-1}, \ldots, x_{n_0}),\ n \geq n_0.$$

We introduce a family groupoids $\{G(m)\}_{m \in \mathbf{N}}$ associated to a uniform space X and a function

$$f : \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$$

having the meaning that for all $(n, n_0) \in \mathbf{N} \times_{\geq} \mathbf{N}$

$$f(n, n_0, x) = f_{n,n_0}(x_{n-1}, x_{n-2}, \ldots, x_{n_0})$$

with $x_{n_0} = x$, or equivalently, $f(n, n_0, x)$ is the state at time n, if at n_0 the state was x.

By a groupoid we mean a set G, together with a partially defined multiplication

$$(\gamma_1, \gamma_2) \to \gamma_1\gamma_2 \ [: G^{(2)} \subset G \times G \to G],$$

and an inverse map

$$x \to x^{-1} \ [: G \to G],$$

satisfying the following properties:

1. $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in G$.
2. If $(\gamma_1, \gamma_2) \in G^{(2)}$ and $(\gamma_2, \gamma_3) \in G^{(2)}$, then $(\gamma_1\gamma_2, \gamma_3), (\gamma_1, \gamma_2\gamma_3) \in G^{(2)}$ and $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$.
3. For all $\gamma \in G$, $(\gamma, \gamma^{-1}), (\gamma^{-1}, \gamma) \in G^{(2)}$, and if $(\gamma, \gamma') \in G^{(2)}$ (respectively, $(\gamma'', \gamma) \in G^{(2)}$), then $\gamma' = x^{-1}(x\gamma')$ (respectively, $\gamma'' = (\gamma''\gamma)\gamma^{-1}$).

For every groupoid G and $\gamma \in G$, we write $r(\gamma) = \gamma\gamma^{-1}$ and $d(\gamma) = \gamma^{-1}\gamma$, and called them the range and respectively, the domain/source of γ. Then $r(G) = d(G)$ is called the unit space of G and denoted $G^{(0)}$. It is helpful to note that $(\gamma_1, \gamma_2) \in G^{(2)}$ if and only if $d(\gamma_1) = r(\gamma_2)$. Two units x, y $\in G^{(0)}$ are equivalent if there is $\gamma \in G$ such that $r(\gamma) = x$ and $d(\gamma) = y$. The sets of units equivalent to x is called the orbit of x and is denoted [x]. If x and y are equivalent units, then

$$G_y^x=\{\gamma \in G: r(\gamma) = x \text{ and } d(\gamma) = y\}.$$

2. A FAMILY OF GROUPOIDS ASSOCIATED TO A DISCRETE PROCESS

In order to define a family of groupoids associated to a process $f : \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$ (that does not necessarily satisfy the semigroup property of a process) we use a uniform structure on X. A uniform space is a set X equipped with a nonempty family $\mathcal{U}$ of subsets of $X \times X$ (called uniform structure on X) satisfying the following conditions:

1. $\Delta \subset U$ for all $U \in \mathcal{U}$, where $\Delta = \{ (x, x) : x \in X \}$.
2. If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.
3. If $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
4. For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $VV \subset U$, where $VV=\{(x, y): \text{there is z such that } (x, z) \in V \text{ and } (z, y) \in V\}$
5. If $U \in \mathcal{U}$, then $U^{-1} = \{(y, x): (x, y) \in U\} \in \mathcal{U}$.

If $\mathcal{U}$ is a uniform structure on X, then $\mathcal{U}$ induces a topology on X. With respect to this topology, $A \subset X$ is open set if and only if for every $x \in A$ there exists $U \in \mathcal{U}$ such that

$\{y: (x, y) \in U\} \subset A$.

The groupoids that are constructed in this section are subgroupoids of $X \times \mathbf{Z} \times X$ viewed as groupoid under the operations:

$(x, k, y)^{-1} =(y, -k, x)$

$(x, k, y) (y, j, z) = (x, k + j, z)$

Proposition 2.1. Let $\mathcal{U}$ be a uniform structure on X, $f : \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$ be a function and m be a nonnegative integer. Let us write

$G(m)(X,\mathcal{U}, f) = \{(x, k, y) \in X \times \mathbf{Z} \times X:$
for all $U \in \mathcal{U}$ there is $n_U \in \mathbf{N}$ such that
for all $n \geq n_U$ we have $n \geq m$, $n+k \geq m$ and
$(f(n+k, m, x), f(n, m, y)) \in U\}$.

Then $G(m)(X,\mathcal{U}, f)$ is a subgroupoid of the trivial groupoid $X \times \mathbf{Z} \times X$.

Proof. Let $(x, k, y) \in G(m)(X,\mathcal{U}, f)$ and let us prove that $(y, -k, x) \in G(m)(X,\mathcal{U}, f)$. Let $U \in \mathcal{U}$. Then $U^{-1} \in \mathcal{U}$, hence there is $n_U \in \mathbf{N}$ such that for all $n \geq n_U$ we have $n \geq m$, $n+k \geq m$ and $(f(n+k, m, x), f(n, m, y)) \in U^{-1}$. Thus for all $n \geq n_U+k$ we have $n-k \geq m$, $n \geq m$ and

$(f(n, m, x), f(n-k, m, y)) \in U^{-1}$.

Consequently, $(f(n-k, m, y), f(n, m, x)) \in U$. Therefore, $(y, -k, x) \in G(m)(X,U, f)$.

Let $((x, k, y), (y, j, z)) \in G(m)(X,\mathcal{U}, f)$ and let us prove that $(x, k+j, z) \in G(m)(X,\mathcal{U}, f)$. Let $U \in \mathcal{U}$. Then there is $V \in \mathcal{U}$ such that $VV \subset U$. Since $(x, k, y) \in G(m)(X,\mathcal{U}, f)$, there is $n_V \in \mathbf{N}$ such that for all $n \geq n_V$, we have $n \geq m$, $n + k \geq m$ and $(f(n+k, m, x), f(n, m, y)) \in V$. Since $(y, j, z) \in G(m)(X,\mathcal{U}, f)$, there is $n'_V \in \mathbf{N}$ such that for all $n \geq n'_V$, we have $n \geq m$, $n + j \geq m$ and $(f(n+j, m, y), f(n, m, z)) \in V$. Thus for all $n \geq \max\{n_V-j, n'_V\}$ we have $n + j \geq m$, $n + j + k \geq m$ and $(f(n+j+k, m, x), f(n+j, m, y)) \in V$.

On the other hand, $n \geq m$, $n + j \geq m$ and $(f(n+j, m, y), f(n, m, z)) \in V$. Therefore

$(f(n+j+k, m, x), f(n, m, z)) \in VV \subset U$ for all $n \geq \max\{n_V-m, n'_V\}$. Hence $(x, k+j, z) \in G(m)(X,\mathcal{U}, f)$.

Definition 2.2. Let $f: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$ be a function and $p \in \mathbf{N}^*$. For each $x \in X$ we write

$$N_{p,f}(x)=\{n_0 \in N: f(n_0 + np, n_0, x) = x \text{ for all } n \in \mathbf{N}\}.$$

An element $x \in X$ is said to be an equilibrium point of the system defined by

$$f: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$$

if $N_{1, f}(x) \neq \emptyset$, i.e. if there is $n_0 \in N$ such that

$$f(n + n_0, n_0, x) = x \text{ for all } n \in \mathbf{N}.$$

An element $x \in X$ is said to be periodic if there is $p \in \mathbf{N}$, $p \geq 2$ such that $N_{p,f}(x) \neq \emptyset$, i.e. if there is $p \in \mathbf{N}$, $p \geq 2$ and there is $n_0 \in N$ such that

$$f(n_0 + np, n_0, x) = x \text{ for all } n \in \mathbf{N}.$$

The smallest p with this property is called period of x.

Definition 2.3. Let X be a space endowed with a uniform structure $\mathcal{U}$. Let us consider a system defined by a function

$$f: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$$

and let $p \in \mathbf{N}$ and $x \in X$ such that $N_{p,f}(x) \neq \emptyset$ (x is equilibrium point or a periodic point). We write

$N_{s,p,f}(x)=\{n_0 \in N_p(x)$: for every $U \in \mathcal{U}$ there is $V_U \in \mathcal{U}$ with the property that if $(x, y) \in V_U$, then $(x, f(n_0 + np, n_0, y)) \in U$ for all $n \in \mathbf{N}\}$

- The point x is said to be stable if $N_{s,p,f}(x) \neq \emptyset$.
- The point x is said to be attractive if there is $n_0 \in N_{p,f}(x_e)$ and there is $U \in \mathcal{U}$ such that for each $(x, y) \in U$ there is $k=k_y \in \mathbf{N}$ with the property that:

$$\lim_{n\to\infty} f(np + k + n_0, n_0, y) = x.$$

- The point x is asymptotically stable if it is stable and attracting.

Let us remark that if x is an equilibrium point of the system defined

$$f: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X,$$

then $G(n_0)(X,\mathcal{U}, f)_x^x = \{x\} \times \mathbf{Z} \times \{x\} \approx \mathbf{Z}$. for all $n_0 \in N_{1,f}(x)$.

Lemma 2.4. Let $\mathcal{U}$ be a uniform structure on X, let

$$f: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$$

be a function, let $p \in \mathbf{N}^*$ and $x \in X$ be such that $N_{p,f}(x) \neq \emptyset$ (x is an equilibrium point or a periodic point). If $n_0 \in N_{p,f}(x)$ and x and y are equivalent units of $G(n_0)(X, \mathcal{U}, f)$, then there is $k \in \mathbf{N}$, $0 \leq k \leq p-1$ such that

$$\lim_{n\to\infty} f(np+k + n_0, n_0, y) = x.$$

If x is an equilibrium point, then if $y \in X$ and

$$\lim_{n\to\infty} f(n, n_0, y) = x,$$

it follows that y and x are equivalent units of $G(n_0)(X, \mathcal{U}, f)$.

Proof. x and y are equivalent units of $G(n_0)(X, \mathcal{U}, f)$ if and only if there is $j \in \mathbf{N}$ such that $(x, j, y) \in G(n_0)(X, \mathcal{U}, f)$ if and only if for all $U \in \mathcal{U}$ there is $n_U \in \mathbf{N}$ such that for all $n \geq n_U$ we have $n \geq n_0$, $n+k \geq n_0$ and

$$(f(n + j, n_0, y), f(n, n_0, x)) \in U.$$

Thus

$$(f(n_0 + np + j, n_0, y), f(n_0 + np, n_0, x)) \in U.$$

Since $n_0 \in N_{p,f}(x)$, it follows that

$$f(n_0 + np, n_0, x) = x.$$

Therefore if x and y are equivalent units of $G(n_0)(X, \mathcal{U}, f)$, then

$$(f(n_0 + np + j, n_0, y), x) \in U.$$

for all $n \geq n_U$ or equivalently,

$$\lim_{n\to\infty} f(np+j + n_0, n_0, y) = x.$$

Let $j = pq + k$, with $q, k \in \mathbf{N}$, $0 \leq k \leq p-1$. Then

$$\lim_{n\to\infty} f((n+q)p+k + n_0, n_0, y) = x.$$

and consequently,

$$\lim_{n\to\infty} f(np+k + n_0, n_0, y) = x.$$

with $0 \leq k \leq p-1$.

If x is an equilibrium point and

$$\lim_{n\to\infty} f(n, n_0, y) = x,$$

then for all $U \in \mathcal{U}$ there is $n_U \in \mathbf{N}$ such that for all $n \geq n_U$ we have

$$(f(n, n_0, y), x) \in U.$$

Since x is an equilibrium point, $f(n, n_0, x) = x$ and therefore for all $n \geq n_U$ we have

$$(f(n, n_0, y), f(n, n_0, x)) \in U.$$

Hence y and x are equivalent units of $G(n_0)(X, \mathcal{U}, f)$.

Proposition 2.5. Let $\mathcal{U}$ be a uniform structure on X, let

$$f: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$$

be a function, let $p \in \mathbf{N}^*$, $x \in X$ be such that $N_{p,f}(x) \neq \emptyset$ and let $n_0 \in N_{p,f}(x)$. If x is in interior of its orbit with respect to the structure of the groupoid $G(n_0)(X, \mathcal{U}, f)$, then x is attractive.

Conversely, if x is attractive, then x is in interior of its orbit with respect to the structure of the groupoid $G(n_0)(X, \mathcal{U}, f_p)$, where

$$f_p: \mathbf{N} \times_{\geq} \mathbf{N} \times X \to X$$

is defined by $f_p(n, n_0, x) = f(np, n_0, x)$ for all $(n, n_0, x) \in \mathbf{N} \times_{\geq} \mathbf{N} \times X$.

Proof. Let us assume that x belongs to the interior of $[x] \subset X \times \mathbf{Z} \times X$. Then there is $U \in \mathcal{U}$ such that $\{y: (x, y) \in U\} \subset [x]$. Let $y \in X$ be such that $(x, y) \in U$. By Lemma 2. 4. since $y \in [x]$, it follows that there is $k \in \mathbf{N}$, $0 \le k \le p-1$ such that

$$\lim_{n\to\infty} f(np+k+n_0,\ n_0, y) = x.$$

Hence x_e is attractive.

Let us assume that x is attractive. Then there is $U \in \mathcal{U}$ such that for each $(x, y) \in U$ there is $k=k_y \in \mathbf{N}$ with the property that:

$$\lim_{n\to\infty} f(np+k+n_0,\ n_0, y) = x.$$

or equivalently,

$$\lim_{n\to\infty} f_p(n + k + n_0, n_0, y) = x.$$

$$\lim_{n\to\infty} f_p(n, n_0, y) = x.$$

On the other hand, if $N_{p,f}(x) \neq \emptyset$, then x is an equilibrium point for the system defined by f_p. Applying Lemma 2.4, it follows that x and y are equivalent units of $G(n_0)(X, \mathcal{U}, f_p)$,. Consequently,

$$\{y: y \in U\} \subset [x].$$

with respect to $G(n_0)(X, \mathcal{U}, f_p)$. Therefore x belongs to the interior of $[x] \subset X$ with respect to $G(n_0)(X, \mathcal{U}, f_p)$.

We have associated a family of groupoids $\{G(m)(X, \mathcal{U}, f)\}_{m\in\mathbf{N}}$ to a discrete system defined by a function

$$f: \mathbf{N} \times_{\ge} \mathbf{N} \times X \to X$$

having the meaning that $f(n, n_0, x)$ is the state at time n, if at n_0 the state was x for all $(n, n_0) \in \mathbf{N} \times_{\ge} \mathbf{N}$ (the state space X is endowed with a uniform structure $\mathcal{U}$). In order to formalize the connections between various "initial moments of time", we use groupoid homorphisms:

$$H : G(m_1)(X, \mathcal{U}, f) \to G(m_2)(X, \mathcal{U}, f)$$

Since the groupoids of the form

$$G(m)(X, \mathcal{U}, f)$$

are subgroupoids of the trivial groupoid

$$X \times \mathbf{Z} \times X,$$

using the results in [2] (Section 2) and those in [3] (Section 2), we can conclude that the groupoid $G(m)(X, \mathcal{U}, f)\}$ can be represented as $\bigcup_{\alpha(x)=\alpha(y)} G_x^y$ with

$$G_y^x = \{(x, nk(\alpha(x)) + k(x) - k(y), y): n\in\mathbf{N}\},$$

where α: $X \to X$ and $k : X\to \mathbf{Z}$ are two functions with following properties:

1. $\alpha(\alpha(x)) =$ for every $x \in X$.
2. $k(\alpha(x)) \ge 0$ for every $x \in X$.
3. If $k(\alpha(x)) \neq 0$ and $x \neq \alpha(x)$, then $k(x) \in \{0, 1, \dots, k(\alpha(x))-1\}$.

For such a groupoid the orbit of $x \in X$ is

$$[x] = \{y: \alpha(x) = \alpha(y)\}.$$

We denote by $\overline{X} = \{[x]: x \in X\}$.

Let us assume that the functions α_1, k_1, respectively, α_2, k_2 describe the groupoid $G(m_1)(X, \mathcal{U}, f)$, respectively $G(m_2)(X, \mathcal{U}, f)$. Applying Corollary 4[4], a homomorphism

$$H : G(m_1)(X, \mathcal{U}, f) \to G(m_2)(X, \mathcal{U}, f)$$

is given by a function $h : X \to X$ with the property that

$$h([x]) \subset [h(x)]$$

for all $x\in X$ and another two functions

$\mu : X \to \mathbf{Z}$

η: $\overline{X} \to \mathbf{Z}$ such that

$$H(x, nk_1(\alpha_1(x))+k_1(x)-k_1(x), y) = (h(x), (n\eta([x])+\mu(x)-\mu(y))k_2(\alpha_2(h(x))) + k_2(h(x))- k_2(h(y)), h(y))$$

for all $(x, nk_1(\alpha_1(x)) + k_1(x) - k_2(y), x)$ in $G(m_1)(X, \mathcal{U}, f)$.

Thus $\{G(m)(X, \mathcal{U}, f)\}_{m\in\mathbf{N}}$ can be viewed as a category. Consequently, we can use the category language for defining, for instance, the equivalence of two nonautonomous systems as the equivalence of the associated groupoid categories.

We can also use other morphisms between groupoids. Let us consider a discrete process f that satisfies the semigroup property:

$$f(n_0, n_0, x) = x \text{ for all } n_0 \in \mathbf{N} \text{ and } x\in X.$$

$$f(n_2, n_0, x) = f(n_2, n_1, f(n_1, n_0, x)) \text{ for all } n_0 \le n_1 \le n_2 \text{ and } x\in X,$$

and let us construct a particular homomorphism for $m_1 \le m_2$:

$$H : G(m_1)(X, \mathcal{U}, f) \to G(m_2)(X, \mathcal{U}, f)$$

$$H(x, k, y) = (f(m_2, m_1, x), k,\ f(m_2, m_1,y))$$

for all $(x, k, y) \in G(m_1)(X, \mathcal{U}, f)$.

Since $f(n+k, m_2, f(m_2, m_1, x)) = f(n+k, m_2, x)$, and $f(n, m_2, f(m_2, m_1, y)) = f(n, m_2, y)$, it follows that H is correctly defined. Also

$$H(r(x, k, y)) = H(x,0,x) = (f(m_2, m_1, x), 0,\ f(m_2, m_1,x)) = r((f(m_2, m_1, x), k,\ f(m_2, m_1,y)))$$

$= r(H(x, k, y))$
$H(d(x, k, y)) = H((y, 0, y)$
$= d(H(y, 0, y))$
$H((x, k, y)(y, j, z)) = H(x , k + j, z) =$
$= (f(m_2, m_1, x), k + z, f(m_2, m_1,z)$
$= (f(m_2, m_1, x), k, f(m_2, m_1,y))$
$(f(m_2, m_1, x), j, f(m_2, m_1,z))$
$= H(x, k, y)H(y, j, z)$

If the discrete system generated by f does not satisfy the semigroup property, we can replace the homorphisms:

$H : G(m_1)(X, \mathcal{U}, f) \to G(m_2)(X, \mathcal{U}, f)$

by a groupoid homomorphism

$H: G[f, m_2](m_1)(X, \mathcal{U}, f) \to G(m_2)(X, \mathcal{U}, f)$,

where $G[f, m_2](m_1)(X, \mathcal{U}, f)$ is a groupoid isomorphic to the groupoid obtained by blowing up the unit space of $G(m_1)(X, \mathcal{U}, f)$ to $f_{m_1,,m_2}(X)$ with

$f_{m_1,,m_2}(x) = f(m_2, m_1, x)$ for all $x \in X$.

More precisely,

$G[f, m_2](m_1)(X, \mathcal{U}, f) = \{ (x, k, y):$
$f(m_2, m_1, x), k, f(m_2, m_1, y) \in G(m_2)(X, \mathcal{U}, f)\}$

and

$H(x, k, y) = (f(m_2, m_1, x), k, f(m_2, m_1,y))$.

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