Z TRANSFORM FOR FRACTIONAL DISCRETE-TIME SYSTEMS WITH MULTIPLICATIVE NOISE AND MEAN SQUARE STABILITY

Ungureanu Viorica Mariela, Assoc. Prof. PhD

"Constantin Brâncuși" University from Târgu Jiu, ROMANIA

ABSTRACT: *The aim of this paper is to give sufficient conditions for the asymptotic stability of a class of linear discrete-time fractional order system with multiplicative noise. Our result is based on a new Z-transform that acts on sequences of random variables with finite mean square.*

KEY WORDS: *linear discrete-time fractional system, asymptotic stability, Z transform, stochastic systems*

1. INTRODUCTION

The theory of fractional systems has recently known a large development due to its numerous applications in many real life processes with infinite delays. We mention here some of their applications in the study of wave propagation in viscoelastic media, dissipation in seismology or in metallurgy [3-4], adaptive and robust control [7-8] etc. Since random factors usually affect the real-world phenomena, the stochastic models prove to be the most suitable solution for their analysis.

For this reason, in this paper we will study the asymptotic properties of a class of autonomous fractional discrete-time linear systems (FDTLSs) affected by sequences of independent real-valued random variables with zero mean and finite variance. In order to solve the stability problems, we define the notion of Z transform for sequences of random variables with finite mean square. By analogy with the classical Z transform, we establish that the new transform inherits many of the well-known properties of the deterministic Z transform. Using this new type of linear transform, we provide sufficient conditions for the mean square asymptotic stability of our FDTL stochastic systems (Theorem 1).

As far as we know, this approach is new.

2. NOTATIONS AND STATEMENT OF THE PROBLEM

Let $(\Omega, \mathbf{F}, P)$ be a probability space. For any integrable, random variable ξ on $(\Omega, \mathbf{F}, P)$, we write $E[\xi]$ for its mean (expectation). Let R and C be the set of real and complex numbers, respectively. We will denote by $L^2(\Omega, \mathrm{R}^d)$, the Hilbert space of all R^d -valued random variables ξ with the property that

$$_2\|\xi\| =^{def} E\left[\|\xi\|^2\right] < \infty .$$

By replacing R^d with C^d in the above definition, we can introduce the Hilbert space $L^2(\Omega, \mathrm{C}^d)$. The product scalar on $L^2(\Omega, \mathrm{C}^d)$ will be denoted by $\langle .,. \rangle_2$ and is defined by $\langle \xi, \eta \rangle_2 = E\langle \xi, \eta \rangle$.

We say that a series $\sum_{n=0}^{\infty}\eta_k$, $(\eta_k)_{k\in\mathbf{N}} \subseteq L^2(\Omega, \mathrm{C}^d)$ is *absolutely convergent* if the numerical series $\sum_{n=0}^{\infty} {}_2\|\eta_k\|$ is convergent.

Discrete-time fractional order systems with stochastic perturbations

Let $\mathrm{R}_+^* = \{x \in \mathrm{R}, x > 0\}$ and $\alpha \in \mathrm{R}_+^*, \alpha < 2$ be fixed. For all $j \in \mathrm{N}$, let $\begin{pmatrix}\alpha \\ j\end{pmatrix}$ denote the generalized binomial coefficient

$$\begin{pmatrix}\alpha \\ j\end{pmatrix} := \begin{cases} 1, j = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha+1-j)}{j!}, j > 0\end{cases}.$$

We consider the Grünwald--Letnikov's definition of the fractional-order operators

$$\Delta^{[\alpha]}x_{k+1} = \frac{1}{h^\alpha}\sum_{j=0}^{k+1}(-1)^j\begin{pmatrix}\alpha \\ j\end{pmatrix}x_{k+1-j},$$

where $h \in \mathrm{R}_+^*$ is the sampling period or time increment.

Let $\{\xi_k\}_{k\in\mathbf{N}} \subseteq L^2(\Omega, \mathrm{R})$ be a sequence of mutually independent random variables on $(\Omega, \mathbf{F}, P)$ satisfying $E[\xi_k] = 0$ and $E[\xi_k^2] = b < \infty$ for all $k \in \mathbf{N}$. We consider the discrete-time fractional system

$$(1) \qquad \begin{aligned} \Delta^{[\alpha]}x_{k+1} &= \mathsf{A}x_k + \xi_k \mathsf{B}x_k, k \in \mathrm{N} \\ x_0 &= x \in \mathrm{R}^d, \end{aligned}$$

where A, B are real d - dimensional matrices. In this the paper, we do not distinguish between the matrix and the linear operator defined by it. As in [9], we have

$$h^\alpha\Delta^{[\alpha]}x_{k+1} = \sum_{j=0}^{k+1}(-1)^j\begin{pmatrix}\alpha \\ j\end{pmatrix}x_{k+1-j}.$$

By multiplying (1) by h^α and by using the last relation, we get

$$x_{k+1} = \left(h^\alpha\mathsf{A} + \alpha I_{\mathrm{R}^d}\right)x_k + \sum_{j=1}^{k}(-1)^j\begin{pmatrix}\alpha \\ j+1\end{pmatrix}x_{k-j} + \xi_k h^\alpha \mathsf{B}x_k.$$

We set $A = h^\alpha\mathsf{A}, B = h^\alpha\mathsf{B}$,

$c_j := (-1)^j\begin{pmatrix}\alpha \\ j+1\end{pmatrix}$, $A_j = c_j I_{\mathrm{R}^d}$, $j \in \mathrm{N}$,

and system (1) can be equivalently rewritten as

$$(2) \quad x_{k+1} = Ax_k + \sum_{i=0}^{k}A_i x_{k-i} + \xi_k Bx_k,$$

$$(3) \quad x_0 = x \in \mathrm{R}^d.$$

Let $\mathbf{F}_n$, $n \in \mathbf{N}^*$ denotes the $\sigma-$ algebra generated by $\{\xi_i, 0 \le i \le n-1\}$. We know (see [9]) that (2)-(3) has a unique solution which belongs to $L^2(\Omega, \mathrm{R}^d)$.

In this paper, we will adopt the following mean square stability notion (see for e.g [1] for deterministic systems).

Definition. *System (2)-(3) is mean square asymptotically (MSA) stable if, for any initial condition* x_0, *we have* $\lim_{k\to\infty} E\|x_k\|^2 = 0$.

2. Z TRANSFORM OF RANDOM SEQUENCES

For any random sequence $\eta = (\eta_k)_{k\in\mathbf{N}} \subseteq L^2(\Omega, \mathrm{R}^d)$, we define its Z transform

$$\mathbf{Z}\{\eta\}(z) = \sum_{k=0}^{\infty}\eta_k z^{-k},$$

for any complex number z for which the above series is absolutely convergent.

It is well known that the ROC (region of convergence) of the above series is either the wide set or the outside of a circle. In the second case, the power series is uniformly convergent in every bounded closed domain lying outside this circle (see for example [6]).

If η is a bounded sequence from $L^2(\Omega, \mathrm{R}^d)$ (i.e there is $M > 0$ such that

$E\|\eta_k\|^2 \le M$ for all $k \in \mathrm{N}$), then the radius of convergence of $\sum_{k=0}^{\infty} \eta_k z^{-k}$ is

$$R = \lim_{k\to\infty} \sqrt[k]{{}_2\|\eta_k\|} \le \lim_{k\to\infty} \sqrt[2k]{M} = 1$$

and $\mathbf{Z}\{\eta\}(z)$ is absolutely convergent for any

$$|z| > 1$$

and uniformly convergent for any $|z| \ge r > 1$.

We often shall use the notation $\mathbf{Z}\{\eta_k\}(z)$ for $\mathbf{Z}\{\eta\}(z)$ to stress the dependence of η on k. Many of the well-known properties of the deterministic Z transform remains valid for random sequences. In what follows, we particularly need the following properties:

Linearity: For any $\eta = (\eta_k)_{k\in\mathbf{N}}$, $\xi = (\xi_k)_{k\in\mathbf{N}} \subset L^2(\Omega, \mathrm{R}^d)$, $a, b \in \mathrm{Z}$ and $z \in$ ROC ($\mathbf{Z}\{\xi\}$) $\cap$ ROC ($\mathbf{Z}\{\eta\}$) we have

$$\mathbf{Z}\{a\xi + b\eta\} = a\mathbf{Z}\{\xi\} + b\mathbf{Z}\{\eta\}.$$

The **proof** follows the same lines as in the case when η and ξ are deterministic and is ommited.

Convolution theorem: Let $x = (x_k)_{k\in\mathbf{N}} \in \mathrm{R}_+$ and $y = (y_k)_{k\in\mathbf{N}} \in L^2(\Omega, \mathrm{R}^d)$ be two causal signals (i.e. $x_k = 0, y_k = 0$ for $k < 0, k \in \mathrm{Z}$), such that $\mathbf{Z}\{\mathbf{x}\}(z)$ and $\mathbf{Z}\{\mathbf{y}\}(z)$ are absolutely convergent for any $|z| > R_1$ and $|z| > R_2$, respectively. Then $(x * y)_k \overset{def}{=} \sum_{i=0}^{\infty} x_{k-i} y_i$, $k \in \mathrm{N}$ is a sequence from $L^2(\Omega, \mathrm{R})$, $\mathbf{Z}(x * y)(z)$ exists for any $|z| > \max\{R_1, R_2\}$ and

$$\mathbf{Z}\{x * y\}(z) = \mathbf{Z}\{x\}(z)\mathbf{Z}\{y\}(z).$$

Proof. It is a direct consequence of Merten's Theorem. Indeed, let

$$X_n = \sum_{k=0}^{n} x_k z^{-k}, Y_n = \sum_{k=0}^{n} y_k z^{-k},$$

$$C_n = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_k y_{k-i} \right) z^{-k} \quad .$$

We know that

$$C_n = x_0 z^{-0}(Y_n - \mathbf{Z}\{y\}) + \ldots$$
$$+ x_n z^{-n}(Y_0 - \mathbf{Z}\{y\}) + X_n \mathbf{Z}\{y\}.$$

Since $X_n \to \mathbf{Z}\{x\}$ as $n \to \infty$, it suffices to prove that $C_n - X_n\mathbf{Z}\{y\} \to 0$ as $n \to \infty$.

It is easy to see that

$$_2\|C_n - X_n\mathbf{Z}\{y\}\| =$$

$$= \sum_{k=0}^{n} \left| x_k z^{-(k)} \right| \ _2\|Y_{n-k} - \mathbf{Z}\{y\}\|$$

$$= \sum_{k=0}^{N} \left| x_k z^{-k} \right| \ _2\|Y_{n-k} - \mathbf{Z}\{y\}\| +$$

$$\sum_{k=N}^{n} \left| x_k z^{-k} \right| \ _2\|Y_{n-k} - \mathbf{Z}\{y\}\|$$

Let $\varepsilon > 0$. Since X_n, Y_n are absolutely convergent, it follows that there is $M > 0$ such that $_2\|Y_n\|, |X_n| < M$ for all $n \in \mathrm{N}$.

On the other hand, X_n is absolutely convergent and, consequently, it is Cauchy, which implies that there is a natural $N_\varepsilon > 0$ such that $\sum_{k=N}^{n} \left| x_{n-k} z^{-(n-k)} \right| < \varepsilon / (2M)$ for all $n \ge N \ge N_\varepsilon$.

Since $Y_n \to \mathbf{Z}\{y\}$ as $n \to \infty$, it follows that there is $N_\varepsilon' > N_\varepsilon > 0$ such that $_2\|Y_n - \mathbf{Z}\{y\}\| < \varepsilon / (2M)$, for all $n \ge N$. Then $_2\|C_n - X_n\mathbf{Z}\{y\}\| \le M\varepsilon / (2M) + M\varepsilon / (2M) = \varepsilon$ for all $n \ge N_\varepsilon'$ and we get the conclusion.

Time Advance

For any $y = (y_k)_{k\in\mathbf{N}} \subset L^2(\Omega, \mathrm{C}^d)$

$$\mathbf{Z}\{y_{n+k})\} = z^k \mathbf{Z}\{y_n\} - y_0 - y_1 z - .. - y_{k-1} z^{k-1}, k \le n$$

Proof. Since

$$\sum_{i=0}^{n} y_{i+k} z^{-i} = z^k \sum_{i=0}^{n} y_{i+k} z^{-i-k} = z^k \sum_{i=k}^{n+k} y_i z^{-i}$$

$$= z^k \sum_{i=0}^{n+k} y_i z^{-i} - y_0 - y_1 z - .. - y_{k-1} z^{k-1},$$

we pass to the limit as n tends to infinity and we get the conclusion.

For any $M > 1$, an M – Stolz sector is the set of complex numbers z with the property that $|z - 1| \le M(|z| - 1)$.

Final value theorem: Assume that $\eta=(\eta_k)_{k\in\mathbf{N}}\subset L^2(\Omega,\mathrm{C}^d)$ is convergent. Then, there is M > 1 such that for any z belonging to the M – Stolz sector, we have

$$\lim_{z\to 1}(z-1)\mathbf{Z}\{\eta\}(z)=\lim_{n\to\infty}\eta_n.$$

All the limits are considered in $_2\|\ \|$.

Proof: Since the ROC of $\mathbf{Z}\{\eta\}$ includes the outside of the unit circle, any M- Stolz sector will be included in the ROC of $\mathbf{Z}\{\eta\}$.

A short computation shows that

$$\text{(5)}\quad \mathbf{Z}\{\eta_{k+1}-\eta_k\}(z)=\lim_{n\to\infty}\sum_{k=0}^{n}(\eta_{k+1}-\eta_k)z^{-k} = (z-1)\mathbf{Z}\{\eta\}(z)-z\eta_0$$

If $\lim_{n\to\infty}\eta_n=\eta\in L^2(\Omega,\mathrm{C}^d)$, then, by the linear property of $\mathbf{Z}$ transform, we have $\mathbf{Z}\{\eta_{k+1}-\eta_k\}(z)=\mathbf{Z}\{\eta_{k+1}\}(z)-\mathbf{Z}\{\eta_k\}(z)$, which implies that $\mathbf{Z}\{\eta_{k+1}-\eta_k\}(z)$ is convergent for $|z|>1$. Abel theorem remains valid for power series with coefficients in a Hilbert space and ensures that, for any z belonging to an M - Stolz sector, we have

$$\lim_{z\to 1}\lim_{n\to\infty}\sum_{k=0}^{n}(\eta_{k+1}-\eta_k)z^{-k}=\sum_{k=0}^{\infty}(\eta_{k+1}-\eta_k) =\lim_{n\to\infty}\eta_n-\eta_0.$$

Passing to the limit in (5), we deduce that

$$\lim_{z\to 1}((z-1)\mathbf{Z}\{\eta\}(z)-z\eta_0)=\lim_{n\to\infty}\eta_n-\eta_0$$

On the other hand

$$\lim_{z\to 1}((z-1)\mathbf{Z}\{\eta\}(z)-z\eta_0)=\lim_{z\to 1}(z-1)\mathbf{Z}\{\eta\}(z)-\eta_0$$

and we obtain the conclusion.

3. APPLICATIONS OF Z TRANSFORM TO STOCHASTIC FRACTIONAL SYSTEM

In this section, we will transform the stochastic system (2)-(3) by applying the above Z transform. For the sake of simplicity, in the following we shall denote by $X(z)$ the Z transform of the sequence x_n representing the solution of (2)-(3).

The *time advance* property ensures that $\mathbf{Z}\{x_{k+1}\}(z)=zX(z)-x_0$.

By assuming that x_k is a causal signal, we see that $\sum_{i=0}^{k}A_i x_{k-i}$ is the k -th term of the convolution product $(\mathbf{A}*x)(k)$ of the sequences $\mathbf{A}=\left((-1)^k\binom{\alpha}{k+1}\right)_{k\geq 0}$ and $(x_k)_{k\geq 0}$.

It is well known (see for example [5]) that

$$\mathbf{Z}\{A_k\}(z)=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k+1}z^{-k} =-z\sum_{k=1}^{\infty}(-1)^k\binom{\alpha}{k}z^{-k}=-z\left[\left(1-\frac{1}{z}\right)^{\alpha}-1\right].$$

From the convolution theorem, it follows that

$$\mathbf{Z}\{\mathbf{A}*x\}(z)=-z\left[\left(1-\tfrac{1}{z}\right)^{\alpha}-1\right]X(z) .$$

In what follows, we assume the following hypothesis:

H2: *the unique solution* x_n *of (2)-(3) form a convergent sequence in* $L^2(\Omega,\mathrm{C}^d)$.

Since $_2\|\xi_n\|=\sqrt{b}$, it follows that $_2\|\xi_n x_n\|$ is a convergent sequence and $\mathbf{Z}\{_2\|\xi_n x_n\|\}(z)$, $\mathbf{Z}\{\xi_n x_n\}(z)$ are well defined on the same ROC as $X(z)$.

By applying the above $\mathbf{Z}$ transform to system (2)-(3), we get

$$zX(z)-x_0=AX(z)-z\left[\left(1-\frac{1}{z}\right)^{\alpha}-1\right]X(z) +B\mathbf{Z}\{\xi_n x_n\}(z)$$

or, equivalently,

$$zX(z)-x_0=AX(z)-z\left(1-\frac{1}{z}\right)^{\alpha}+ zX(z)+B\mathbf{Z}\{\xi_n x_n\}(z).$$

It follows that $X(z)$ satisfies the equation

$$-x_0 = AX(z) - z\left(1-\frac{1}{z}\right)^{\alpha} X(z) + B\mathbf{Z}\{\xi_n x_n\}(z) \Leftrightarrow$$

$$-x_0 = \left(A - z\left(1-\frac{1}{z}\right)^{\alpha}\right) X(z) + B\mathbf{Z}\{\xi_n x_n\}(z) \Leftrightarrow$$

(6) $\quad -x_0 = A^{\alpha}(z) X(z) + B\mathbf{Z}\{\xi_n x_n\}(z),$

where $A^{\alpha}(z)$ denotes the matrix $A - z\left(1-\frac{1}{z}\right)^{\alpha} I_d$.

4. APPLICATIONS TO MEAN SQUARE ASYMPTOTIC STABILITY

In this section, we shall apply Z transform properties to obtain sufficient conditions for the asymptotic stability of the FDTLS (2)-(3).

Theorem 1. *Assume that the solution* x_n *of the* FDTLS (2)-(3) *is* ${}_2\|\ \|$ *- convergent, all the roots of the equation*

$$\det A^{\alpha}(z) = 0$$

are strictly inside the unit circle and

(7) $\qquad \left\|A^{\alpha}(1)^{-1}B\right\|\sqrt{b} < 1.$

Then $\lim_{n\to\infty} x_n = 0,$ *i.e the system (2)-(3) is mean square asymptotically stable.*

Proof. Since all the roots of the equation $\det\left(A^{\alpha}(z)\right) = 0$ are inside the unit circle, it follows that $A^{\alpha}(z)$ is invertible for any z belonging to an M- Stolz sector, M > 1. By multiplying (6) with $(z-1)A^{\alpha}(z)^{-1}$, we get

$$(z-1)A^{\alpha}(z)^{-1}x_0 + (z-1)X(z)$$
$$= (z-1)A^{\alpha}(z)^{-1}\mathrm{B}\mathbf{Z}\{\xi_n x_n\}(z)$$

for any complex number z. We take the norm and we get

$${}_2\left\|(z-1)A^{\alpha}(z)^{-1}x_0 + (z-1)X(z)\right\| =$$
$${}_2\left\|(z-1)A^{\alpha}(z)^{-1}B\mathbf{Z}\{\xi_n x_n\}(z)\right\|$$
$$\le \left\|A^{\alpha}(z)^{-1}B\right\|\left|(z-1)\right|\mathbf{Z}\{{}_2\|\xi_n x_n\|\}(|z|)$$
$$\le \left\|A^{\alpha}(z)^{-1}B\right\|\sqrt{b}\left|(z-1)\right|\mathbf{Z}\{{}_2\|x_n\|\}(|z|)$$
$$\le \left\|A^{\alpha}(z)^{-1}B\right\|\sqrt{b}M\left(|z|-1\right)\mathbf{Z}\{{}_2\|x_n\|\}(|z|).$$

Applying the final *value theorem* and passing to the limit $z \to 1$ in the above inequality (for z belonginig to an M- Stolz), we have

$${}_2\left\|\lim_{n\to\infty} x_n\right\| \le \left\|A^{\alpha}(1)^{-1}B\right\|\sqrt{b}M \lim_{n\to\infty} {}_2\|x_n\|.$$

From (7) and the following equality ${}_2\left\|\lim_{n\to\infty} x_n\right\| = \lim_{n\to\infty} {}_2\|x_n\|$, we deduce that

$$\lim_{n\to\infty 2}\|x_n\|\left(1 - \left\|A^{\alpha}(1)^{-1}B\right\|\sqrt{b}M\right) \le 0,$$

which is equivalent to $\lim_{n\to\infty} {}_2\|x_n\| = 0$. The conclusion follows.

In the absence of the stochastic perturbation, we get the following result, which is known for deterministic systems.

Corollary 1. *Assume that* $B_n = 0, n \in N$ *and all the elements of the matrix* $A^{\alpha}(z)$ *are rational functions. Then, the system (2)-(3) is asymptotically stable if the roots of the equation*

$$\det A^{\alpha}(z) = 0$$

are strictly inside the unit circle.

Proof. Since the system is causal, i.e $x_k = 0$ for $k < 0, k \in Z$, we can use the hypotheses and the inverse discrete-time Fourier transform to deduce that the solution x_n of the FDTLS (2)-(3) is ${}_2\|\ \|$ - convergent. The conclusion follows from the above theorem.

We note that the unique solution x_n of the deterministic system FDTLS (2)-(3) can be written as $x_n = F(n, n_0)x_0$, where $F(n, n_0)$ is a linear and bounded operator on R^d. The assumption that $\lim_{n\to\infty} x_n = 0,$ implies the strong convergence of the sequence $F(n, n_0)$. In finite dimensions, $F(n, n_0)$ is also norm convergent and the asymptotic property defined in this paper ensures both the stability and attractivity properties defined in [5] for a fractional deterministic system. It follows that our notion of asymptotic stability implies the one from [5].

Consequently, if the equation

$$\det A^{\alpha}(z)=0$$

has a root z such that $|z|>1$, then $\lim_{n\to\infty} x_n$ cannot be 0 as it follows from [5]. Therefore, the condition

C: *all the roots of the equation*

$$\det A^{\alpha}(z)=0$$

are inside the unit circle

is necessary for the asymptotic stability of the deterministic FDTLS (2)-(3).

5. CONCLUSIONS

In this paper, we provide sufficient conditions for the mean square asymptotic stability of an autonomous LDTFSs with multiplicative noise. Unlike the deterministic case, where C is a necessary condition for the asymptotic stability, in the stochastic case this conditions may not be sufficient. Further investigations concerning Z transform for sequences of random variables could clarify this issue and could also lead to improved results. With the exception of the $_2\| \|$-convergence of the solution x_n of the FDTLS (2)-(3), the assumptions of Theorem 1 are easy to verify. However, to establish some sufficient conditions for the $_2\| \|$ - convergence of x_n is still a challenge problem in the stochastic framework.

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