SEMI-INVARIANT SUBMANIFOLDS IN METRIC GEOMETRY OF AFFINORS

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ABSTRACT: We introduce a generalization of structured manifolds as the most general Riemannian metric g associated to an affinor (tensor field of (1, 1)-type) F and initiate a study of their semi-invariant submanifolds. These submanifolds are generalizations of CR- submanifolds of almost complex geometry and semi-invariant submanifolds of several interesting geometries (almost product, almost contact and others). We characterize the integrability of both invariant and anti-invariant distribution.

KEY WORDS: Riemannian geometry; CR-submanifolds, semi-invariant submanifold;

1. INTRODUCTION

The geometry of manifolds endowed with geometrical structures has been intensively studied and several important results have been published, see Yano-Kon [14]. The more important classes of such manifolds are formed by almost complex, almost product, almost contact, almost paracontact manifolds for which the cited book offers a good introduction. The geometry of submanifolds in these manifolds is very rich and interesting, as well, see for example the classical [7] or the more recent survey [8]. CR-submanifolds introduced by Bejancu in [2] (for almost complex geometry) respectively [5] (for almost contact geometry) had a great impact on the developing of the theory of submanifolds in these ambient manifolds; a proof of this fact is given by the books [4] and [13].

In the present paper we first introduce the concept of $(g, F, +1)$-manifold which contains as particular cases all the above types of structures. Then we study semi-invariant sub-manifolds of a $(g, F, +1)$-manifold, which are extensions of CR-submanifolds to this general class of manifolds. We find necessary and sufficient conditions for the integrability of both distributions on a semi-invariant submanifold, see Theorems 3.1 and 3.3.

2. METRIC GEOMETRY OF AFFINORS AND SUBMANIFOLDS

Let M be an $m-$dimensional manifold for which we denote by $C^\infty(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $C^\infty(M)-$module of smooth sections of the tangent bundle TM of M ; let $X, Y, Z, \ldots$ denote such vector fields. We use the same notation $\Gamma(V)$ for any other vector bundle V over M. Let also $\mathcal{T}_1^1 M$ be the $C^\infty(M)-$module of $\Gamma(TM \otimes T^*M)$ i.e. the real space of tensor fields of $(1,1)-$type on M. Let consider a fixed $F \in \mathcal{T}_1^1 M$ usually called affinor ([9]) or vector $1-$form; the remarkable affinor of every manifold is the Kronecker tensor field $I = (\delta_j^i)$.

Let now g be a Riemannian metric on M.

Definition 2.1. M is a $(g, F, +1)-$*manifold* if:

$$g(FX, Y) + g(X, FY) = 0. \qquad \textbf{(1)}$$

The geometry of the data (M, g, F, +1) is called *affinor-metric geometry*. If in particular, F_x is nondegenerate at any point $x \in M$ then we say that M is a *nondegenerate* $(g, F, +1)-$manifold; otherwise, M is called degenerate $(g, F, +1)-$manifold.

The relation (1) says that the $g-$adjoint of F is $F^* = -F$. In literature there is an abundance of examples of $(g, F, +1)$ −manifolds. Some of the main examples are presented here:

Examples 2.2.

1. An *almost Hermitian manifold* ([4]) (M, g, J) is a nondegenerate $(g, F, +1)-$ manifold.
2. An *almost parahermitian manifold* ([1]) (M, g, P) is a nondegenerate $(g, F, +1)$ −manifold.
3. An *almost contact metric manifold* ([4]) (M, g, ϕ, ξ, η) is a $(g, F, +1)$ −manifold; as $\phi(\xi) = 0$, M is degenerate.
4. An *almost paracontact manifold* ([12]) (M, g, ϕ, ξ, η) is a $(g, F, +1)$ −manifold. As in the previous example we have $\phi(\xi) = 0$ and therefore M is degenerate.
5. The general case of a nondegenerate $(g, F, +1)$ −manifold is called *structured manifold* in [11].

Recall that a real $2m$ −dimensional manifold M is called an *almost symplectic manifold* if it is endowed with a nondegenerate 2 −form $\Omega \in \Lambda^2(M)$. We derive the following characterization:

Proposition 2.3 Let M be a $(g, F, +1)$ −manifold. Then M is nondegenerate if and only if Ω defined by:

$$\Omega(X, Y) = g(FX, Y) \qquad \textbf{(2)}$$

is an almost symplectic structure. In this case m is even.

Proof. **Ω** is skew-symmetric. A straightforward computation yields that Ω is nondegenerate if and only if M is a nondegenerate $(g, F, +1)$ −manifold. ■

Example 2.4. For Example 1.2.1 Ω is exactly the *fundamental* or Kähler 2 −form and then inspired by this fact we introduce:

Definition 2.5. For a nondegenerate $(g, F, +1)$ −manifold Ω is call *the fundamental* 2 *−form*.

In the last part of this section let us recall briefly the geometry of Riemannian submanifolds. Consider an n −dimensional submanifold N of M. Then the main objects induced by the Levi-Civita connection $\tilde{\nabla}$ of (M, g) on N are involved in the well known Gauss-Weingarten equations:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \qquad \textbf{(3)}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$. Here ∇ is the Levi-Civita connection on N , h is the second fundamental form of N, A_V is the Weingarten operator with respect to the normal section V and $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}N$ of N. Let us point out that h and A_V are related by:

$$g(h(X, Y), V) = g(A_V X, Y). \qquad \textbf{(4)}$$

If h vanishes identically on N then N is called *totally geodesic.*

3. SUBMANIFOLDS IN AFFINOR-METRIC GEOMETRY

Next, we consider a submanifold N of a $(g, F, +1)$ −manifold M. Then g induces a Riemannian metric on N which we denote by the same symbol g. Then, following the definition given by Bejancu [2] for CR-submanifolds we introduce a special class of submanifolds of M as follows:

Definition 3.1. N is a *semi-invariant submanifold* of M if there exists a distribution D on N satisfying the conditions:

(i) D is F −invariant:

$$F(D_x) \subset D_x, \forall x \in N. \qquad \textbf{(5)}$$

(ii) The complementary orthogonal distribution $D^{\perp}$ to D in TN is F −anti-invariant, that is:

$$F(D_x^{\perp}) \subset T_x^{\perp}N, \forall x \in N. \qquad \textbf{(6)}$$

(iii) $F^2(D^{\perp})$ is a distribution on N .

Some particular classes of semi-invariant submanifolds are defined as follows. Let p and q be the ranks of the distributions D and $D^{\perp}$ respectively. If $q = 0$, that is $D^{\perp} = \{0\}$, we say that N is an F *−invariant submanifold* of M. If $p = 0$, that is $D = \{0\}$, we call N an F *−anti-invariant submanifold* of M.

If $pq \neq 0$ then N is called a *proper* semi-invariant submanifold. Now, we denote by $\widetilde{D}$ the complementary orthogonal vector bundle to $F(D^{\perp})$ in $T^{\perp}N$. If $\widetilde{D} = \{0\}$ then we say that N is a *normal* F −semi-invariant submanifold. Thus, N is an F −invariant, respectively F −anti-invariant, if and only if:

$$F(TN) \subset TN \text{ (resp. } F(TN) \subset T^{\perp}N). \quad \textbf{(7)}$$

N is normal F −semi-invariant if and only if:

$$F(D^{\perp}) = T^{\perp}N. \quad \textbf{(8)}$$

Examples 3.2.
1) For Example 2.2.1 we obtain the classical concept of CR −submanifold of Bejancu [4]; the condition iii) is satisfied from $J^2 = -I$.
2) For Example 2.2.2 we obtain the notion of semi-invariant submanifold; for the almost parahermitian case see [1] while for the second case see [3]. The condition iii) is satisfied again from $P^2 = -I$.
3) For Example 2.2.3 we obtain the notion of semi-submanifold [4] with $\xi \in T^{\perp}N$. This last condition implies $TN \subset ker\, \eta$ and since $\phi|_{ker\, \eta}$ is an almost complex structure we get iii).
4) For Example 2.2.4 we obtain the concept of semi-submanifold from [10] with $\xi \in T^{\perp}N$. Again this condition means $TN \subset ker\, \eta$ and since $\phi|_{ker\, \eta}$ is an almost product structure we have iii).
5) The condition (iii) does not appears in [11].

Returning to the Definition 3.1 we deduce that the tangent bundle TN and the normal bundle $T^{\perp}N$ of a semi-invariant submanifold N have the orthogonal decompositions:

$$TN = D \oplus D^{\perp}, T^{\perp}N = F(D^{\perp}) \oplus \widetilde{D}. \quad \textbf{(9)}$$

Then we denote by P and Q the projection morphisms of TN on D and $D^{\perp}$ respectively and obtain for $X = PX + QX \in \Gamma(TN)$:

$$FX = \varphi X + \omega X \quad \textbf{(10)}$$

where we put:

$$\varphi = F \circ P, \ \omega = F \circ Q. \quad \textbf{(11)}$$

Thus φ is a tensor field of $(1, 1)$ −type on N while ω is a $F\,(D^{\perp})$ −valued vector 1 −form on N . Thus we derive:

Proposition 3.3. Let N be a semi-invariant submanifold of a $(g, F, +1)$ −manifold M. Then:
(iv) N is a $(g, \varphi, +1)$ −manifold.
(v) $F^2(D^{\perp})$ is a vector subbundle of $D^{\perp}$.
(vi) The vector bundle $\widetilde{D}$ is F −invariant i.e. for all $x \in N$ we have: $F(\widetilde{D}x) \subset \widetilde{D}x$.

Proof. (iv) By definition, g is a Riemannian metric on N and φ is a tensor field of (1, 1)-type on N; we need only to show **(5)**. By using **(1)** for F we obtain for $X, Y \in \Gamma(TN)$:

$$g(\varphi X, Y) = g(FPX, Y) = g(FPX, PY) = -g(PX, FPY) = = -g(X, FPY) = -g(X, \varphi Y).$$

(v) Take $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$ in **(5)**: $g(X, F^2Y) = -g(FX, FY) = 0$ since $FX \in \Gamma(D)$ and $FY \in \Gamma(T^{\perp}N)$. Hence $F^2(D^{\perp})$ is orthogonal to D and by condition (iii) we deduce that $F^2(D^{\perp})$ is a vector subbundle of $D^{\perp}$.
(vi) Take $X \in \Gamma(TN), Y \in \Gamma(D^{\perp})$ and $V \in \Gamma(\widetilde{D})$. Then we obtain:

$$g(FV, X) = -g(V, FX) = -g(V, \varphi X + \omega X) = 0$$

and:

$$g(FV, FY) = -g(V, F^2Y) = 0$$

since $\varphi X \in \Gamma(D), \omega X \in \Gamma(FD^{\perp})$ and $F^2Y \in \Gamma(D^{\perp})$. Thus $F\widetilde{D}$ is orthogonal to $TN \oplus FD^{\perp}$, that is $F\widetilde{D}$ is a vector subbundle of $\widetilde{D}$. This completes the proof of the proposition. ■

In the non-degenerated case we have equalities for the above inclusions:

Corollary 3.4. Let N be a semi-invariant submanifold of a nondegenerate $(g, F, +1)$ −manifold M . Then:
1) the above distributions satisfy:

$$F\,(D) = D, \ F^2(D^{\perp}) = D^{\perp}, \ F(\widetilde{D}) = \widetilde{D}. \quad \textbf{(12)}$$

2) $D^{\perp}$ and $F\,(D^{\perp})$ are Lagrangian distribution on (TM, Ω). In particular if N is a normal semi-

invariant submanifold then $T^{\perp}N$ is a Lagrangian submanifold in (TM, Ω).
Proof. We need to prove only 2).
2.1) Let $X, Y \in \Gamma(D^{\perp})$; then $\Omega(X, Y) = g(FX, Y) = 0$ since $FX \in \Gamma(T^{\perp}N)$ while $Y \in \Gamma(TN)$.
2.2) Let $X, Y \in \Gamma(F(D^{\perp}))$; then $\Omega(X, Y) = g(FX, Y) = 0$ since $FX \in \Gamma(TN)$ while $Y \in \Gamma(T^{\perp}N)$. ■

4. INTEGRABILITY OF DISTRIBUTIONS ON A SEMI-INVARIANT SUBMANIFOLD

Let N be a semi-invariant submanifold of a $(g, F, +1)$ −manifold M. Then we recall that the Nijenhuis tensor field of F is defined as follows ([4]):

$$N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY], \quad \textbf{(13)}$$

for any $X, Y \in \Gamma(TM)$. In a similar way, the Nijenhuis tensor field of φ on N is given by:

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \quad \textbf{(14)}$$

for any $X, Y \in \Gamma(TN)$. We recall that a tensor field of $(1,1)$ −type defines an integrable structure on a manifold if and only if its Nijenhuis tensor field vanishes identically on the manifold. Now we obtain necessary and sufficient conditions for the integrability of D and $D^{\perp}$in terms of Nijenhuis tensor fields of F and φ.
Theorem 4.1. Let N be a semi-invariant submanifold of a $(g, F, +1)$ −manifold M. Then the following assertions are equivalent:
1) D is an integrable distribution.
2) The Nijenhuis tensor field of φ satifies:

$$Q \circ N_\varphi = 0, \ \forall X, Y \in \Gamma(D). \quad \textbf{(15)}$$

3) The Nijenhuis tensor fields of F and φ satisfy the equality: $N_F = N_\varphi$ on D.
Proof. Firstly, we note that D is integrable if and only if:
$Q([X, Y]) = 0, \ \forall X, Y \in \Gamma(D)$. **(16)**

Since the last three terms in the right side of (14) lie in $\Gamma(D)$ we deduce that:

$$Q \circ N_\varphi(X, Y) = Q([FX, FY]), \forall X, Y \in \Gamma(D). \quad \textbf{(17)}$$

As M is nondegenerate we deduce that φ is an automorphism on $\Gamma(D)$. Thus the equivalence of 1) and 2) follows directly. Next, we obtain for any $X, Y \in \Gamma(D)$:

$$N_F(X, Y) = N_\varphi(X, Y) + F\omega([X, Y]) - \omega([\varphi X, Y]) - \omega([X, \varphi Y]). \quad \textbf{(18)}$$

If D is integrable then the last three terms of (18) vanishes and this yields 3). Conversely, suppose that $N_F = N_\varphi$ on D; then:

$$F\omega([X, Y]) = \omega([\varphi X, Y] + [X, \varphi Y]). \quad \textbf{(19)}$$

Obviously the right-hand-side of the previous equation is in $\Gamma(F(D^{\perp})) \subset \Gamma(T^{bot}N)$. On the other hand, the left-hand-side is in $\Gamma(F^2 D^{\perp}) \subset \Gamma(TN)$; we conclude that both sides in (19) must vanish.
Finally, from: $F^2 Q([X, Y]) = 0$ and F^2 automorphism of $\Gamma(TM)$ we deduce 1). ■

Remark 4.2. For Example 1.2.1 the equivalence of 1) and 2) is exactly the Theorem 2.2. of [4] while the equivalence of 1) and 3) is the Theorem 2.1. of [4].
Now, we consider $X, Y \in \Gamma(D^{\perp})$. Then taking into account that $\varphi X = \varphi Y = 0$ we get:

$$N_\varphi(X, Y) = F^2 P[X, Y] \quad \textbf{(20)}$$

and this enables us to state the following:
Theorem 4.3. Let N be a semi-invariant submanifold of a nondegenerate $(g, F, +1)$ −manifold. Then $D^{\perp}$ is integrable if and only if the Nijenhuis tensor field of φ vanishes identically on $D^{\perp}$.
Remark 4.4. For Example 1.2.1 the above result is the Theorem 2.3. of [4].

5. CONCLUSION

We can connect our study with the almost symplectic geometry and this fact opens some

possible further applications in physical sciences having as an example the relationship between CR-structures and Relativity pointed out in the last Chapter of [4].
Also the second part of the above Corollary 3.4. is extremely important since it relates the geometry of semi-invariant submanifolds with the almost symplectic geometry, a topic very studied from the point of view of applications in Analytical Mechanics.

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