SOME CONSIDERATIONS REGARDING ANALITICAL SOLVING OF THE REYNOLD’S EQUATION FROM FACE SEAL

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Abstract: Taking into consideration a mechanical face seal which runs in hydrodynamic duty, we intend to understand what’s going on under its surfaces. Until nowadays the experimental study to measure the parameters of film between two seal surfaces is very difficult. To explain these phenomena from this interface is necessary to know the theoretical pressures. The paper uses a primary seal model where the seal surfaces are considered with misalignment but the ring centre distance is constant. Considering some simplified hypothesis it determines the Reynold’s equation, it’s solving leading to the pressure equation of sealing interface.

Keywords: pressure, hydro-dynamic, thickness.

1. Introduction

The mechanical face seals are used to obtain the seals from the mechanical systems, whose good running depends, in a critical manner, of the mechanical seal, even if the cost is minimum related to the whole system.

This fact has determined the physicians, the mathematicians and the engineers to be more interest of these mechanical face seals performances. The thick films approximation, introduced by O. Reynolds in 1886 is the fundamental base of the theoretical studies in this domain.

There are two criteria of good running for the mechanical face seals: the sealing film stability and the mobile primary mechanical seal equilibrium. The equilibrium is complete when the resultant of forces given by the pressure HD or HS in the film is equal with the load of the exterior forces.

A well knowledge of the pressure field is necessary in all the experimental, theoretical or numerical studies. Until now, the experimental study is very difficult because the measurements done in the film which separates the two surfaces are very approximately. This is due to the complex phenomena which take place and have very different natures (HD, THD, elastic, etc)

We admit that this thing cannot be always accepted, that the correct running of a mechanical face seal is linked by a complete fluid film existence in the $S_1 – S_2$ interface (fig.1).

![Fig. 1 The representation of the primary seal.](image-url)
Also, we say that the mechanical seal runs in a hydro-dynamic duty. A mixed duty, different from the hydro-dynamic one and characterized by some consistent contacts existence between the surfaces, may exist if:
- the exterior forces that tend to eliminate the film are important;
- the fluid vaporizes itself;
- the surfaces geometry isn’t favorable for apparition and maintaining the fluid film.

2. Reynold’s Equation

We study the movement of a viscid fluid between two surfaces, in the following hypothesis:
- The fluid is Newtonian and the flow is laminar;
- The exterior mass forces and the inertial ones are insignificant;
- The film thickness is very small related to the other dimensions of the sealing space;
- The fluid speed on the film height is very small related to the components in the other two directions; the speeds proportion is the same with the proportion of the corresponding dimensions;
- The surfaces asperity heights are smaller comparing with the film thickness

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \rho v_r \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \rho v_\theta \right) + \frac{\partial(v_r)}{\partial z} + \frac{\partial p}{\partial t} = 0
\]  

(1)

and

\[
\frac{\partial p}{\partial r} = \frac{\partial}{\partial z} \left[ \frac{1}{\rho} \frac{\partial v_r}{\partial z} \right], \quad \frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{\partial}{\partial z} \left[ \frac{1}{\rho} \frac{\partial v_\theta}{\partial z} \right], \quad \frac{\partial p}{\partial z} = 0
\]  

(2)

where: \( r, \theta, z \) are the cylindrical coordinates; \( v_r, v_\theta, v_z \) are the speed components in the cylindrical coordinates; \( p \) is the pressure, \( \rho \) is the specific consistency; \( \mu \) is the viscosity coefficient.

On \( S_1 \) and \( S_2 \) surfaces (by equation \( z = H_1 \) and \( z = H_2 \)) we put the following conditions at limit:

\[
v_r = v_{r1}; \quad v_\theta = v_{\theta1}; \quad v_z = v_{z1}, \text{ pentru } z = H_1(r, \theta, t)
\]

\[
v_r = v_{r2}; \quad v_\theta = v_{\theta2}; \quad v_z = v_{z2}, \text{ pentru } z = H_2(r, \theta, t)
\]  

(3)

We reduce the problem (1), (2), (3) to one problem for the unknown \( p \).

From (2) results \( p=p(r, \theta, t) \). We consider \( \mu=\mu(r, \theta, t) \) and then (2) \( 1,2 \) becomes:

\[
\frac{\partial^2 v_r}{\partial z^2} = \frac{1}{\mu} \frac{\partial p}{\partial r} \quad \text{and} \quad \frac{\partial^2 v_\theta}{\partial z^2} = \frac{1}{\mu r} \frac{\partial p}{\partial \theta}
\]  

(4)

The obtained system integrates itself twice related to \( z \), and taking into account that \( p(dr) \) and \( \frac{\partial p}{\partial r} \) do not depend on \( z \), it results:

\[
v_r = \frac{1}{\mu} \frac{\partial p}{\partial r} \frac{z^2}{2} + C_1 z + C_3 \quad \text{and} \quad v_\theta = \frac{1}{\mu r} \frac{\partial p}{\partial \theta} \frac{z^2}{2} + C_2 z + C_4
\]  

(5)

Replacing in (5) the constants, it results:
\[ v' = \frac{1}{2 \mu} \frac{\partial p}{\partial r} \left[ z^2 - z(H_1 + H_2) + H_1 H_2 \right] + \frac{v'_1 - v'_2}{H_1 - H_2} (z - H_1) + v'_1 \]  \hspace{1cm} (6)

\[ v'^o = \frac{1}{2 \mu r} \frac{\partial p}{\partial \theta} \left[ z^2 - z(H_1 + H_2) + H_1 H_2 \right] + \frac{v'^o_1 - v'^o_2}{H_1 - H_2} (z - H_1) + v'^o_1 \]  \hspace{1cm} (7)

We integrate the continuity equation (1) related to \( z \) on the range \([H_1, H_2]\) and we obtain

\[ \int_{H_1}^{H_2} \frac{\partial}{\partial r} (\rho v') \, dz + \int_{H_1}^{H_2} \frac{\partial}{\partial \theta} (\rho v'^o) \, dz + \int_{H_1}^{H_2} r \frac{\partial}{\partial z} (\rho v^z) \, dz + \int_{H_1}^{H_2} r \frac{\partial \rho}{\partial t} \, dz = 0 \]  \hspace{1cm} (8)

Taking into account the derivation formula of the integral and considering \( \rho = \rho (r, \theta, t) \), then (8) becomes

\[ \frac{\partial}{\partial r} \int_{H_1}^{H_2} (\rho v') \, dz - \frac{\partial H_2}{\partial r} \rho v'_z + \frac{\partial H_1}{\partial r} \rho v'_r + \frac{\partial}{\partial \theta} \int_{H_1}^{H_2} (\rho v'^o) \, dz - \frac{\partial H_2}{\partial \theta} \rho v'^o_z + \frac{\partial H_1}{\partial \theta} \rho v'^o_r + r \rho (v'_z - v'^o_z) + \]

\[ + r \frac{\partial \rho}{\partial t} (H_2 - H_1) = 0 \]

In order to calculate the above integrals, we take (6) and (7) into account and it results:

\[ \frac{\partial}{\partial r} \left[ r (H_2 - H_1)^3 \frac{\partial \rho}{\partial r} \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{\mu r} (H_2 - H_1)^3 \frac{\partial \rho}{\partial \theta} \right] = 6 \rho (H_2 - H_1) \frac{\partial}{\partial r} [v'_1 + v'_2] + 6 \rho \frac{\partial (H_2 - H_1)}{\partial r} + \]

\[ + 6 (H_2 - H_1) \frac{\partial}{\partial \theta} [(v'_1 + v'_2)] + 6 \rho (v'_1 - v'_2) \frac{\partial (H_2 + H_1)}{\partial \theta} + + 12 (v'_1 - v'_2) \]  \hspace{1cm} (9)

2.1. Particularly case – face seal

We suppose that \( S_1 \) and \( S_2 \) are coaxial circular crowns, \( S_1 \) is fixed; \( S_2 \) has a rotation movement, with angular speed \( \omega \), around the Oz axis (fig. 2).

In order to find out \( H_1 = H(r, \theta, t) \), or \( M_1(r, \theta, z) \in S_1 \). We notice that \( M_1 \) and \( N_1 \) have the same \( z \).

Because \( \chi_1 \) is small, we approximate \( \tan \chi_1 = \sin \chi_1 \approx \chi_1 \), so:

\[ H_1(r, \theta, t) = OO_1 + r \chi_1 \sin \theta \]  \hspace{1cm} (10)

Similarly for \( H_2 \) (but we replace \( \theta \) with \( \theta - \omega t \) because the \( S_2 \) movement), we obtain:

\[ H_2(r, \theta, t) = OO_1 + h_0 + r \chi_2 \sin (\theta - \omega t) \]

Due to the fact that the fluid is consistent, it adheres to the surfaces, so:

\[ v'_1 = 0, \quad v'_2 = 0, \quad v'_1 = 0, \quad v'_2 = r \omega, \quad v'_2 = 0 \]

We consider \( h_0 \) a constant value, so we can suppose that \( v'_2 = 0 \). Equation (9) becomes

\[ \frac{\partial}{\partial r} \left[ \frac{r}{\mu} (H_2 - H_1)^3 \frac{\partial \rho}{\partial r} \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{\mu r} (H_2 - H_1)^3 \frac{\partial \rho}{\partial \theta} \right] = - 6 \rho (H_2 - H_1) \frac{\partial^2}{\partial r \partial \theta} \]  \hspace{1cm} (11)

We note \( h = H_2 - H_1 \) and consider \( \mu \) as a constant value. Equation (11) becomes:

\[ \frac{\partial}{\partial r} \left( rh \frac{\partial \rho}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( h \frac{\partial \rho}{\partial \theta} \right) = 6 \mu \nu r^2 \left[ \chi_2 \cos (\theta - \omega t) + \chi_1 \cos \theta \right] \]  \hspace{1cm} (12)
Fig. 2. The Mathematic Model of the Non-alignment Mechanical Face Seal

We introduce the a-dimensional measures:

- The thickness $\tilde{h} = \frac{h}{h_0}$, The radius $\tilde{r} = \frac{r}{R_e}$; $\tilde{R} = \frac{R_i}{R_e}$, the relative pitch $\chi = \frac{X_1}{X_2}$

  - the pitching – film thickness parameter $e_2 = \frac{R_e}{h_0} \chi$
  
  - the angular position of the rotor $\Omega = \omega r$
  
  - the pressure $\overline{p} = \frac{h_0^2}{6\mu R_e^2\omega} p$.

Then, for $r \in [R_i, R_e]$ we have $\tilde{r} \in [\tilde{r}_1, 1]$, and $\frac{\partial}{\partial r} = \frac{\partial}{\partial \tilde{r}} \frac{\partial \tilde{r}}{\partial r}$, so $\frac{\partial}{\partial r} = \frac{1}{R_e \frac{\partial \tilde{r}}{\partial r}}$.

The a-dimensional film thickness becomes:

$\overline{h} = 1 + \tilde{e} \tilde{e}, [\sin(\theta - \Omega) - \chi \sin \theta]$, and the Reynolds equation in a-dimensional measures:

$$\frac{\partial}{\partial \tilde{r}} \left[ \tilde{h}^3 \frac{\partial \overline{p}}{\partial \tilde{r}} \right] + \frac{\partial}{\partial \theta} \left[ \tilde{h}^3 \frac{\partial \overline{p}}{\partial \theta} \right] = -e_2 \tilde{e}^2 [\cos(\theta - \Omega) + \chi \cos \theta]$$  \hspace{1cm} (13)

3. Integration of Reynold’s Equation  

In order to simplify the notations, instead of $\overline{h}, \overline{r}, \overline{p}$ we will use $h, r$ and $p$. Because $e_2 < 1$ (from $R_e(\chi + \chi) < h_0$, so $\theta + \chi < 1$) we will develop in ranges of power after $e_2$ the functions $h_1$ and $p$. We have:

$$h = 1 + e_2 h_1(r, \theta), \quad cu \ h_1(r, \theta) = \left[ \sin(\theta - \Omega) - \chi \sin \theta \right]$$

$$p = p_0(r, \theta) + e_2 p_1(r, \theta) + e_2^2 p_2(r, \theta) + .......$$  \hspace{1cm} (14)
We introduce the relations (14) in (13) and we obtain:

\[
\frac{\partial}{\partial r} \left[ r(1+e_2h_1)^3 \frac{\partial}{\partial r} \left( \sum_{k=0}^{\infty} e_2^k p_k \right) \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{r} (1+e_2h_1)^3 \frac{\partial}{\partial \theta} \left( \sum_{k=0}^{\infty} e_2^k p_k \right) \right] = -e_2r^2[\cos(\theta - \Omega) + \chi \cos \theta]
\]

Solving this equation and identifying the \( e_2^k \) coefficients, we keep in mind the first two equations:

\[
r \frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial^2 p_1}{\partial \theta^2} + \frac{\partial^2 p_1}{\partial r^2} = -r^2[\cos(\theta - \Omega) + \chi \cos \theta] \] (15)

\[
r \frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial^2 p_2}{\partial \theta^2} + \frac{\partial^2 p_2}{\partial r^2} = -\left[ 3 h_1 + 3r \frac{\partial h_1}{\partial r} \right] \frac{\partial p_1}{\partial r} - 3 \frac{\partial h_1}{\partial \theta} \frac{\partial p_1}{\partial \theta} - \frac{3}{r} h_1 \frac{\partial^2 p_1}{\partial \theta^2} \] (16)

Because \( p(r, \theta) = p(r, \theta + 2\pi) \) it is expected like \( p_1(r, \theta) = p_1(r, \theta + 2\pi) \) so we can solve in Fourier range \( p_1 \) and \( p_2 \) after \( \theta \). After identifying the trigonometrically functions coefficients we obtain equations which are particular cases of the Euler type equation. In the end it results:

\[
p_1(r, \theta) = -\frac{1}{8} \left[ r^2 - (r_1^2 + 1)r + r_1^2 \right] [\cos \Omega + \chi \cos \theta + \sin \Omega \sin \theta] \] (17)

\[
p_2(r, \theta) = \frac{1}{32} \chi \sin \Omega \left[ 1 - 2r_1^2 + (r_1^4 - 1) \frac{\ln r}{\ln r_1} + 2(r_1^2 + 1)r^2 - 3r^4 \right] + \frac{1}{32} \sin 2\Omega \frac{5 + 8r_1^2 + 5r_1^4}{1 + r_1^2} r^2
\]

\[
\quad - 2 \frac{r_1^4}{1 + r_1^2} \frac{1}{r^2} - 3r_1^2 - 5r_1^4 \] \cos 2\theta +
\]

\[
+ \frac{1}{32} \left( \cos 2\Omega - \chi^2 \right) \left[ 5 + 8r_1^2 + 5r_1^4 \frac{1}{1 + r_1^2} r^2 - 2 \frac{r_1^4}{1 + r_1^2} \frac{1}{r^2} + 3r_1^2 + 5r_1^4 \right] \sin 2\theta \] (18)

From (14), (17) and (18) we assume:

\[
\begin{align*}
 p(r, \theta) &= e_2 p_1(r, \theta) + e_2^2 p_2(r, \theta) = -e_2^2 \left[ r^2 - (r_1^2 + 1)r + r_1^2 \right] . \\
 \cdot [\cos \Omega + \chi \cos \theta - \sin \Omega \sin \theta] &+ \frac{e_2^2}{32} \left[ \chi \sin \Omega \left[ 1 - 2r_1^2 + (r_1^4 - 1) \frac{\ln r}{\ln r_1} + 2(r_1^2 + 1)r^2 - 3r^4 \right] + \\
&+ \left[ \sin 2\Omega \cos 2\Omega - (\cos 2\Omega - \chi^2) \sin 2\Omega \right] \frac{5 + 8r_1^2 + 5r_1^4}{1 + r_1^2} r^2 - 2 \frac{r_1^4}{1 + r_1^2} \frac{1}{r^2} - 3r_1^2 - 5r_1^4 \right] \end{align*}
\]

We return to the notations \( \bar{p}, \bar{r}, \bar{r}_1 \) and we write:

\[
\begin{align*}
 \bar{p} (\bar{r}, \theta) &= -e_2^2 \left[ \bar{r}^2 - (\bar{r}_1^2 + 1)\bar{r} + \bar{r}_1^2 \right] \left[ \cos \Omega + \chi \cos \theta - \sin \Omega \sin \theta \right] + \\
&+ \frac{e_2^2}{32} \left[ \chi \sin \Omega \left[ 1 - 2\bar{r}_1^2 + (\bar{r}_1^4 - 1) \frac{\ln \bar{r}}{\ln \bar{r}_1} + 2(\bar{r}_1^2 + 1)\bar{r}^2 - 3\bar{r}^4 \right] + \left[ \sin 2\Omega \cos 2\Omega - (\cos 2\Omega - \chi^2) \sin 2\Omega \right] \frac{5 + 8\bar{r}_1^2 + 5\bar{r}_1^4}{1 + \bar{r}_1^2} \bar{r}^2 - 2 \bar{r}_1^4 \frac{1}{1 + \bar{r}_1^2} \frac{1}{\bar{r}^2} - 3\bar{r}_1^2 - 5\bar{r}_1^4 \right] \end{align*}
\]

The relation (19) represents a solution of the Reynolds equations for the non-dimensional pressure.
4. Conclusions

Many papers have shown the presence of a fluid film between the two surfaces of the primary mechanical seal. Different hydro-dynamic bearing mechanisms have been proposed in order to explain the creation and the maintaining of this film. There is no one explanation and lots of hypotheses were released. Lots of theoretical papers which have been considered the distance between the axis constant, also have demonstrated that the film bearing varies generally in time.

This paper too shows that the relation (1) is not respected. So, the two mechanical seal surfaces may approach and bear away during the rotation. We conclude that the thickness at the center of the sealing space varies in time.

References