A SURVEY M-POLAR FUZZY GRAPHS

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Abstract. I will begin with the presentation of the basic definitions required for the development of this survey. Rosenfeld [17] first introduced the concept of fuzzy graphs. After that fuzzy graph theory becomes a vast research area. In this article, I wanted to survey these types of graphs. Ghorai, G., and M. Pal [10],[11],[12] defined m-polar fuzzy graphs using m-polar fuzzy set. Neighborhood degree of a vertex v, degree of a vertex v and the closed degree of a vertex v are defined.

Keywords: fuzzy graphs, bipolar fuzzy graphs, m-polar fuzzy graphs, weak isomorphism, neighborhood degree of a vertex v, degree of a vertex, closed degree of a vertex v, regularity, m-polar ψ-morphism in product mFG.

1. Introduction


This study is organized as follows: In Section 1, introduction is given and the literature review is illustrated. Section 2 represents a brief study of some graph theoretic concept used in this paper. In Section 3, the notion of m-polar ψ-morphism is introduced on product mFG as a generalization of our usual homomorphism. The action of this morphism is studied and established some results on weak and co-weak isomorphism. d2-degree and total d2-degree of a vertex in product mFGs are defined and studied their properties. Section 4 represents the conclusion of the paper.

2. Basic Definitions

Let V=\{x_0, x_1, ..., x_n\} a some set to be called the set of vertices. Let Γ be the multivalued application of the V set in itself, Γ : V → P(V), P(V) - set of parts V. It's called graph pair G=(V, Γ). Graph G is attached to diagram constructed: each vertices x, ∈V is represented by a point and the connection between two vertices x_i, x_j ∈ V It is represented by a segment oriented from x_i to x_j, if x_j ∈ Γ(x_i). Ordered pair (x_i, x_j) is called edge and has x_i as the initial extremity and x_j as the final extremity. There is another way to define a graph, namely,
using the set of vertices V and the array of edges, which we will note with \( E : G = (V, E) \). We say the two edges of the graph are adjacent if they have a common extremity. The degree of a vertex in G is the number of edges incident with the vertex.

A fuzzy set \( A \) on a set \( X \) is characterized by a mapping \( \mu : X \rightarrow [0,1] \) which is called the membership function. A fuzzy set is denoted by \( A = \{ (x, \mu_A(x)) | x \in X \} \). Let \( V \) be a finite set nonempty.

A fuzzy graph (FG) \([17]\) is a pair \( G : (V, \sigma, \mu) \) where \( \sigma \) is a fuzzy subset of a set \( V \) \( \sigma : V \rightarrow [0,1] \) and \( \mu \) is a fuzzy relation on \( \sigma \), \( \mu(x,y) \leq \sigma(x) \land \sigma(y) \) for all \( x, y \in V \). \( \mu \) is reflexive and symmetric \([17]\).

[5] A fuzzy graph \( H : (\tau, \nu) \) is called a partial fuzzy sub graph of \( G : (\sigma, \mu) \) if \( \tau(u) \leq \sigma(u) \) for every \( u \) and \( \nu(u, v) \leq \mu(u, v) \) for every \( u \) and \( v \). \([19]\) In particular we call a partial fuzzy subgraph \( H : (\tau^*, \nu^*) \) a fuzzy subgraph of \( G : (\sigma, \mu) \) if \( \tau^*(u) = \sigma(u) \) for every \( u \) in \( \tau^* \) and \( \nu^*(u, v) = \mu(u, v) \) for every arc \((u, v)\) in \( \nu^* \).

Through the \([0,1]^m \) set, where \( m \) is an natural number, we will understand a set endowed with a relationship of order “\( \leq \)” dened by \( x \leq y \), for each \( i = 1,2,\ldots,m \); \( p_i(x) \leq p_i(y) \) where \( x, y \in [0,1]^m \) and \( p_i : [0,1]^m \rightarrow [0,1] \) is the \( i \)-th projection mapping.

[22] Let \( X \) be a non-empty set. A bipolar fuzzy set \( E \) on \( X \) is an object having the form \( E = \{(x, \mu^x_E(x), \mu^N_E(x)) | x \in X \} \), where \( \mu^x_E : X \rightarrow [0,1] \) denotes a positive membership degree of the elements of \( X \) and \( \mu^N_E : X \rightarrow [-1,0] \) denotes a negative membership degree of the elements of \( X \).

[2] By a bipolar fuzzy graph of a graph \( G^* = (V,E) \) is a pair \( G = (V,A,B) \) where \( A : V \rightarrow [0,1] \), \( A = (\mu^p_A, \mu^N_A) \) is a bipolar fuzzy set in \( V \) and \( B = (\mu^p_B, \mu^N_B) \) is a bipolar relation on \( V \), such that \( \mu^p_b(x,y) \leq \min \{ \mu^p_A(x), \mu^N_A(y) \} \) and \( \mu^N_b(x,y) \geq \max \{ \mu^N_A(x), \mu^N_A(y) \} \) for all \((x,y) \in E \). We call \( A \) the bipolar fuzzy vertex set of \( V \), \( B \) the bipolar fuzzy edge set of \( E \), respectively.

[6] An \( m \)-polar fuzzy set, (or a \([0,1]^m \)-set) on \( X \) is a mapping \( A : X \rightarrow [0,1]^m \). The set of all \( m \)-polar fuzzy sets on \( X \) is denoted by \( m(X) \).

In the following \( G^* \) represents a crisp graph and \( G = (V,A,B) \) represents a product \( m \)FG of \( G^* \).

3. \( m \)-polar fuzzy graphs

[9] A product \( m \)-polar fuzzy graph of a graph \( G^* = (V,E) \) is a pair \( G = (V,A,B) \) where \( A : V \rightarrow [0,1]^m \) is an \( m \)-polar fuzzy set in \( V \) and \( B : \mathcal{G} \rightarrow [0,1]^m \) is an \( m \)-polar fuzzy set in \( \mathcal{G} \) such that \( p_i \circ B(x,y) \leq p_i \circ A(x) \times p_i \circ A(y) \) for all \((x,y) \in \mathcal{G}^i \), \( i = 1,2,\ldots,m \) and \( \mathbf{B}(x,y) = 0 \) for all \((x,y) \in \mathcal{G} \), \( (0 = (0,0,\ldots,0) \) is the smallest element in \([0,1]^m \).

[10] \( G \) is called strong if \( p_i \circ B(x,y) = p_i \circ A(x) \times p_i \circ A(y) \) for all \((x,y) \in \mathcal{G}^i \), \( i = 1,2,\ldots,m. \) \( G \) is called complete if \( p_i \circ B(x,y) = p_i \circ A(x) \times p_i \circ A(y) \) for all \((x,y) \in E \), \( i = 1,2,\ldots,m. \)
[10] Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product mFGs of the graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively.

A weak isomorphism between $G_1$ and $G_2$ is a bijective mapping $\phi: V_1 \rightarrow V_2$ such that $\phi$ is a homomorphism and $p_i \circ A_i(x_i) = p_i \circ A_2(\phi(x_i))$ for all $x_i \in V_1$ for $i = 1, 2, ..., m$.

A co-weak isomorphism between $G_1$ and $G_2$ is a bijective mapping $\phi: V_1 \rightarrow V_2$ such that $\phi$ is a homomorphism and $p_i \circ B_i(x_i, y_i) = p_i \circ B_2(\phi(x_i, y_i))$ for all $(x_i, y_i) \in V_i$ for $i = 1, 2, ..., m$.

Ghorai, G., and M. Pal, introduced in [9] the notions of neighborhood degree of a vertex $v$, degree of a vertex $v$ and the closed degree of a vertex $v$:

[10] Let $G = (V,E)$ product mFG of the graphs $G^* = (V,E)$.

(i) The neighborhood degree of a vertex $v$ is defined as

$$d_N(v) = (d^1_N(v), d^2_N(v), ..., d^m_N(v))$$

where $d^i_N(v) = \sum_{u \in N(v)} p_i \circ A(u), i = 1, 2, ..., m$.

(ii) The degree of a vertex $v$ in $G$ is defined by $d_G(v) = (d^1_G(v), d^2_G(v), ..., d^m_G(v))$ where

$$d^i_G(v) = \sum_{u \in V \cap (v, e) E} p_i \circ B(u,v), i = 1, 2, ..., m.$$ If all the vertices of $G$ have same degree, then $G$ is called regular product mFG.

(iii) The closed degree of a vertex $v$ is defined by $d^c_G(v) = (d^1_G[v], d^2_G[v], ..., d^m_G[v])$ where

$$d^i_G[v] = d^i_G(v) + p_i \circ A(v), i = 1, 2, ..., m.$$ If each vertex of $G$ has same closed degree, then $G$ is called totally regular product mFG.

Also in [10] are introduced the notions of $m$-polar $\psi$-morphism in product mFG, $d_2$-degree, total $d_2$-degree, $(2; \overline{k})$-regularity and totally $(2; \overline{k})$-regularity:

The $d_2$-degree of a vertex $u$ in $G$ is $d_2(u) = (d^1_2(u), d^2_2(u), ..., d^m_2(u))$ where $d^i_2(u) = \sum_{u \in V \cap (v, e) E} p_i \circ B^2(u,v)$, is such that $p_i \circ B^2(u,v) = \sup \{ p_i \circ B(u,u_i) \land p_i \circ B(u_i,v) \}, i = 1, 2, ..., m$.

The minimum $d_2$-degree of $G$ is denoted as $\delta_2(G) = (\delta^1_2(G), \delta^2_2(G), ..., \delta^m_2(G))$ where

$$\delta^i_2(G) = \land \{ d^i_2(u) \mid u \in V \} .$$

The maximum $d_2$-degree of $G$ is denoted as $\Delta_2(G) = (\Delta^1_2(G), \Delta^2_2(G), ..., \Delta^m_2(G))$ where

$$\Delta^i_2(G) = \lor \{ d^i_2(u) \mid u \in V \} .$$

If $d_2(u) = \overline{k}$ for all $u \in V$ then $G$ is said to be $(2; \overline{k})$-regular product mFG.

The total $d_2$-degree of a vertex $u \in V$ is defined as

$$td_2(u) = (td^1_2(u), td^2_2(u), ..., td^m_2(u))$$

where $td^i_2(u) = \sum_{u \in V \cap (v, e) E} p_i \circ B^2(u,v) + A(u)$.

Theorem 3.1. [10] Let $G = (V,A,B)$ be a product mFG. Then $A(u) = \overline{c} = (c_1, c_2, ..., c_m)$ for all $u \in V$ if and only if the following are equivalent:

(i) $G$ is a $(2; \overline{k})$-regular product mFG.
(ii) $G$ is a totally $(2; \tilde{k} +c)$- regular product $mFG$.

[10] Let $G_1 = (V_1, A_1, B_1)$ and $G_2 = (V_2, A_2, B_2)$ be two product $mFGs$. Then a bijective function $\psi : V_1 \rightarrow V_2$ is called an $m$-polar morphism or $m$-polar $\psi$ - morphism if there exist positive real numbers $k_1, k_2$ such that for $i = 1,2,\ldots,m$

(i) $p_i \circ A_2(\psi(u)) = k_1 p_i \circ A_1(u)$ for all $u \in V_1$.

(ii) $p_i \circ B_2(\psi(u), \psi(v)) = k_2 p_i \circ B_1((u, v))$ for all $(u, v) \in \bar{V}_1^2$.

In this case, $\psi$ is called $(k_1, k_2)$ $m$-polar $\psi$ - morphism from $G_1$ to $G_2$. If $k_1 = k_2 = k$, then we call an $m$-polar $k$-morphism. When $k = 1$, we obtain usual $m$-polar morphism.

**Theorem 3.2.** [10] The relation $\psi$ - morphism is an equivalence relation in the collection of all product $mFGs$.

**Theorem 3.3.** [10] Let $G_1$ and $G_2$ be two product $mFGs$ and $\psi$ be a $(k_1, k_2)$ $m$-polar fuzzy morphism from $G_1$ to $G_2$ for some non-zero $k_1$ and $k_2$. Then the image of strong edges in $G_1$ are also strong edges in $G_2$ if and only if $k_1 = k_2$.

**Theorem 3.3.** [10] If the product $mFG$ $G_1$ is co-weak isomorphic to the product $mFG$ $G_2$ and $G_1$ is regular, then $G_2$ is regular also.

**Remark 3.4.** [10] If the product $mFG$ $G_1$ is co-weak isomorphic to $G_2$ and $G_1$ is strong, then $G_2$ need not be strong.

**Theorem 3.5.** [10] Let $G_1$ and $G_2$ be two product $mFGs$. If $G_1$ is weak isomorphic to $G_2$ and $G_1$ is strong, then $G_2$ is also strong.

**Corollary 3.6.** [10] Let $G_1$ and $G_2$ be two product $mFGs$. If $G_1$ is weak isomorphic to $G_2$ and $G_1$ is regular, then $G_2$ need not be regular.

**Theorem 3.7.** [10] If the product $mFG$ $G_1$ is co-weak isomorphic with a strong regular product $mFG$ $G_2$, then $G_1$ is strong regular product $mFG$.

**Theorem 3.8.** [10] Let $G_1$ and $G_2$ be two isomorphic product $mFGs$. Then $G_1$ is strong regular if and only if $G_2$ is strong regular.

**Theorem 3.9.** [10] A strong $mFG$ $G$ is strong regular if and only if its complement $\bar{G}$ is strong regular.

4. Conclusions

The fuzzy graph theory is one of the most developing area of research. This paper presents the basic definitions and some properties of product $m$-polar fuzzy graph introduced by [10]. A product $mFG$ gives more precision compared to the fuzzy and bipolar fuzzy models.
REFERENCES