ASPECTS REGARDING REDUCTION OF A RECTILINEAR HOMOGENEOUS BAR IN A MATERIAL POINT SYSTEM HAVING THE SAME INERTIAL PROPERTIES

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ABSTRACT: *The paper presents aspects regarding reduction calculation of a rectilinear homogenous bar in a material point system maintaining the same inertial properties as the bar*

KEYWORDS: reduction, homogeneous rectilinear bar, inertial properties;

INTRODUCTION

The dynamic behaviour of a rigid in movement is conditioned by three categories of mechanical magnitudes, called mechanical characteristics, namely: inertial characteristics of the rigid; kinetic characteristics of a rigid; dynamic characteristics.

The studies effected on mechanical processes led to the conclusion that the movement of a rigid, in interaction with other bodies in the medium, is influenced both by the quantity of substances contained, and by the way of the substance distribution in the volume occupied by the rigid.

The inertial characteristics of a rigid, which characterizes the inertia phenomenon of the substance, are represented by two categories of mechanical magnitudes: the mass of the rigid, which characterizes the quantity of substance contained in the volume occupied by the rigid; the inertia momentums, which are used to characterize the way in which the substance in a rigid is distributed, with their help the inertia of a body in rotation movement being expressed. The need of simplification of mathematical calculations, and the desire of more general approaches, led to models being adopted that might replace the real bodies, so that the studies would not show great discrepancies compared to the real cases.

BAR REDUCTION IN THE MATERIAL POINTS SYSTEM

It has been considered that from the point of view of the calculations of the inertia momentums and the centrifugal momentums, the bodies can be classified in four categories:

- bodies of 0 order of magnitude , are bodies that can be reduced to a point or a system of material points (a point size is zero);
- bodies of 1 order of magnitude, are bodies that can be reduced to a line segment , to a curve arch or a complex curve(a curve has only one dimension – they are also called bars;
- bodies of 2 order of magnitude, are bodies that can be reduced to a plane surface, a curved surface or a complex surface (a surface has two dimensions) – they are also called plates;
- bodies of 3 order of magnitude, are real homogeneous bodies(with volume and mass) that can have exterior surfaces plane or curved surface (volume has three dimensions).

Bodies that that have only right angles and plane faces are called simple bodies.

Simple elementary bodies are bodies that by addition or extraction can make any simple body of the same class.

As in forces reduction that pursues replacement of any number of forces and couple of forces in only two magnitudes, which forms the reduction torque, similarly, it is desired to reduce a body of a superior order to one of “0” order. All reductions propose bodies having the same mass, the same mass centre and the same inertia matrix as the reduced element.

As it has been shown, in *order 1* category of bodies, all homogeneous bodies that can be perfectly determined, by using only one parameter, are found. As a rule, these bodies are are called bars and their only dimension is length. The straight bar can be considered the only simple elementary body known.

According to the reduction theorem, *any straight bar can be reduced to a system made up of three material bodies to which the bar mass is assigned, like this; the end of the bar will be assigned one sixth (1/6)of the bar mass, and the mass centre of the bar the rest of two thirds (2/3).*

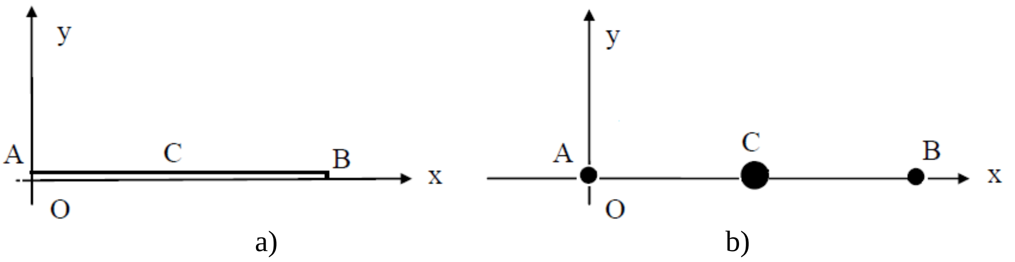

Fig. 1. Homogeneous straight bar situated on Ox axis, in xOy plane.
a) as continuous medium; b) as reduced to a system of material points

To demonstrate this theorem, since it is known that the mass of a homogeneous body is invariant to the reference system, the rectilinear homogeneous bar is considered $AB = l$ ($A \equiv O$), mass m of (Fig. 1.a), for which the position of the mass centre is known:

$$x_C = \frac{l}{2};\ y_C = 0 \tag{1}$$

And inertia momentums to end A and to C mass centre:

$$\begin{aligned} J_A = J_O &= \frac{ml^2}{3}, \\ J_C &= \frac{ml^2}{12} \end{aligned} \tag{2}$$

For the system model of material points of Fig. 2b, proposed to be reduced, due to the synetry and homogeneousness of the bar, the masses of points A and B will be considered equal ($m_A = m_B$) , and equal to m_1 , and C point mass equal to m_2. Since the total mass of the point system should be equal to that of the bar, it results that:

$$m = 2m_1 + m_2; \Rightarrow m_2 = m - 2m_1 \tag{3}$$

For the model proposed in Fig. 2b, applying the formula (3), we set out the condition that the inertia momentum of the material points system to point C should be equal with the inertia momentum to C mass centre of the bar, hence:

$$J_C = m_A\left(\frac{l}{2}\right)^2 + m_B\left(\frac{l}{2}\right)^2 \Rightarrow \frac{ml^2}{12} = 2m_1\left(\frac{l}{2}\right)^2 \Rightarrow$$
$$\Rightarrow \frac{ml^2}{12} = 2\frac{ml^2}{4} \Rightarrow m_1 = \frac{m}{6} \quad (4)$$

From (13) it results:

$$m_2 = m - 2\frac{m}{6} = m - \frac{m}{3} = \frac{2m}{3} \quad (5$$

In order to verify the results the expression of the inertia momentum to point $A \equiv O$ for the proposed material point system model is used, to verify whether it coincides with the one calculated for the homogeneous straight bar.

$$J_A = J_O \equiv m_C\left(\frac{l}{2}\right) + m_B l^2 \Rightarrow \frac{ml^2}{3} \equiv \frac{2m}{3}\left(\frac{l}{2}\right)^2 + \frac{m}{6}l^2 \Rightarrow$$
$$\Rightarrow \frac{ml^2}{3} \equiv \frac{ml^2}{6} + \frac{ml^2}{6} \Rightarrow \frac{ml^2}{3} \equiv \frac{2ml^2}{6} \Rightarrow \frac{ml^2}{3} \equiv \frac{ml^2}{3} \Rightarrow \quad (6)$$
$$\Rightarrow 0 \equiv 0$$

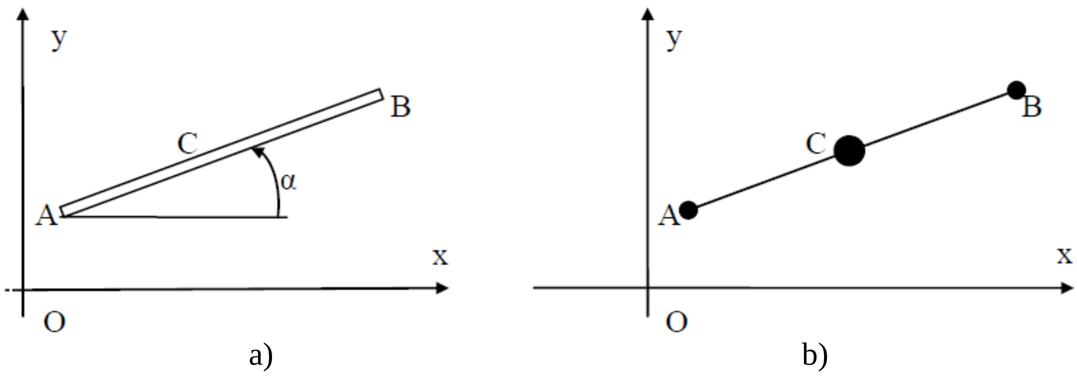

Fig. 2. Homogeneous straight bar situated in any position in xOy plane
a) as continuous medium; b) as system of material points

The same results have been obtained in the case of reduction of homogeneous straight bar as well, situated in xOy plane, defined by points $A(x_A, y_A)$ and $B(x_B, y_B,)$, Fig. 2, the bar mass, m, and its length, l, being known.

The bar position being known, given by the angle with the horizontal direction, α, as well as the bar ends coordinates, shown in Fig. 3, and considering the following equations:

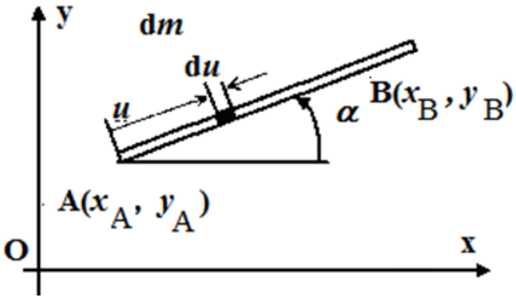

Fig. 3. Calculation model for homogeneous straight bar in plane xOy

$$\rho = \frac{m}{l} = \frac{dm}{du}; x_B = x_A + l\cos\alpha; y_B = y_A + l\sin\alpha; l\cos\alpha = x_B - x_A; l\sin\alpha = y_B - y_A \quad (7)$$

Non-null inertia momentums will have the following formulae:

$$\begin{aligned} J_x &= \int (y_A + u\sin\alpha)^2 dm = \rho\int_0^l (y_A + u\sin\alpha)^2 du = \\ &= m\left(y_A^2 + y_A l\sin\alpha + \frac{l^2}{3}\sin^2\alpha\right) = \frac{m}{3}\left(3y_A^2 + 3y_A l\sin\alpha + l^2\sin^2\alpha\right) = \\ &= \frac{m}{3}\left[3y_A^2 + 3y_A(y_B - y_A) + (y_B - y_A)^2\right] = \\ &= \frac{m}{3}\left(3y_A^2 + 3y_A y_B - 3y_A^2 + y_B^2 + y_A^2 - 2y_A y_B\right) = \frac{m}{3}\left(y_A^2 + y_A y_B + y_B^2\right) \end{aligned} \quad (8)$$

$$\begin{aligned} J_y &= \int (x_A + u\cos\alpha)^2 dm = \rho\int_0^l (x_A + u\cos\alpha)^2 du = \\ &= m\left(x_A^2 + x_A l\cos\alpha + \frac{l^2}{3}\cos^2\alpha\right) = \frac{m}{3}\left(3x_A^2 + 3x_A l\cos\alpha + l^2\cos^2\alpha\right) = \\ &= \frac{m}{3}\left[3x_A^2 + 3x_A(x_B - x_A) + (x_B - x_A)^2\right] = \\ &= \frac{m}{3}\left(3x_A^2 + 3x_A x_B - 3x_A^2 + x_B^2 + x_A^2 - 2x_A x_B\right) = \frac{m}{3}\left(x_A^2 + x_A x_B + x_B^2\right) \end{aligned} \quad (9)$$

$$\begin{aligned} J_z &= J_x + J_y = m\left(x_A^2 + y_A^2 + x_A l\cos\alpha + y_A l\sin\alpha + \frac{l^2}{3}\right) = \\ &= \frac{m}{3}\left(x_A^2 + y_A^2 + x_A x_B + y_A y_B + x_B^2 + y_B^2\right) \end{aligned} \quad (10)$$

$$
\begin{aligned}
J_{xy} &= \int (x_A + u\cos\alpha)(y_A + u\sin\alpha)dm = \rho \int_0^l (x_A + u\cos\alpha)(y_A + u\sin\alpha)du = \\
&= m\left(x_A y_A + x_A \frac{l}{2}\sin\alpha + y_A \frac{l}{2}\cos\alpha + \frac{l^2}{3}\sin\alpha\cos\alpha \right) = \\
&= m\frac{6x_A y_A + 3x_A(y_B - y_A) + 3y_A(x_B - x_A) + 2(y_B - y_A)(x_B - x_A)}{6} = \\
&= m\frac{(6x_A y_A + 3x_A y_B - 3x_A y_A + 3y_A x_B - 3y_A x_A}{6} + \\
&+ \frac{2y_B x_B - 2y_B x_A - 2y_A x_B + 2y_A x_A}{6}\Bigg) = \frac{m}{3}\left(x_A y_A + x_B y_B + \frac{x_A y_B + x_B y_A}{2} \right)
\end{aligned}
\tag{11}
$$

For the model proposed in Fig. 2b, we set the condition that the axial inertia momentum of the points system, J_x,to Ox should be equal to the axial inertia momentum of the bar to the same axis Ox, and due to the bar symmetry and homogeneousness, it will be considered that the masses of points A and B are equal, and equal with m_1, and the mass of C point equal with m_2, hence:

$$
m = 2m_1 + m_2 ; \Rightarrow m_2 = m - 2m_1 \tag{12}
$$

For results verification, the formulae of the axial inertia momentums J_y to Oy axis are used, the centrifugal inertia momentum J_{xy} as well as the calculation equation of the weight centre, all for the proposed model of the material points system, to see whether they coincide with those calculated for homogeneous straight bar:

$$
\begin{aligned}
J_y &= \frac{m}{6}x_A^2 + \frac{m}{6}x_B^2 + \frac{2m}{3}\left(\frac{x_A + x_B}{2}\right)^2 = \\
&= \frac{m}{6}\left(x_A^2 + x_B^2\right) + \frac{2m}{3}\cdot\frac{1}{4}\left(x_A^2 + x_B^2 + 2x_A x_B\right) = \\
&= \frac{2mx_A^2 + 2mx_B^2 + 2x_A x_B}{6} = \frac{m}{3}\left(y_A^2 + y_B^2 + y_A y_B\right)
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
J_{xy} &= \frac{m}{6}x_A y_A + \frac{m}{6}x_B y_B + \frac{2m}{3}\left(\frac{x_A + x_B}{2}\right)\left(\frac{y_A + y_B}{2}\right) = \\
&= \frac{m(x_A y_A + x_B y_B)}{6} + \frac{2m}{3}\cdot\frac{1}{4}(x_A y_A + x_B y_B + x_A y_B + x_B y_A) = \\
&= m\left(\frac{2x_A y_A + 2x_B y_B + x_A y_B + x_B y_A}{6}\right) = \\
&= m\left(\frac{x_A y_A + x_B y_B}{3} + \frac{x_A y_B + x_B y_A}{6}\right)
\end{aligned}
\tag{14}
$$

$$x_C = \frac{\frac{m}{6}x_A + \frac{2m}{3}\left(\frac{x_A + x_B}{2}\right) + \frac{m}{6}x_B}{\frac{m}{6} + \frac{m}{6} + \frac{2m}{3}} =$$
$$= \frac{\frac{m}{6}(x_A + x_B) + \frac{m}{3}(x_A + x_B)}{m} = \frac{x_A + x_B}{2}$$
$$y_C = \frac{\frac{m}{6}y_A + \frac{2m}{3}\left(\frac{y_A + y_B}{2}\right) + \frac{m}{6}y_B}{\frac{m}{6} + \frac{m}{6} + \frac{2m}{3}} = \qquad (15)$$
$$= \frac{\frac{m}{6}(y_A + y_B) + \frac{m}{3}(y_A + y_B)}{m} = \frac{y_A + y_B}{2}$$

From the above it is seen that the proposed model for the replacement of the bar (reduction of the bar) has all non-null axial inertia momentums and centrifugal inertia momentums equal with those of the given bar. Similarly, the coordinates of the weight centre of the material points system are equal with those of the bar.

CONCLUSIONS

Body reduction operation refers to replacement of a body, which has a certain degree of complexity, with a less complex body, but which has the same specific properties as the initial one, more exactly, the same inertia properties.

The models used in classical mechanics are just that, body reductions, with the following reduction criteria: the reduced model has a shape resembling the the body; the reduced model has momentum of the order of 1 (static momentum, implicitly mass centre) identical with those of the body.

The reduced model should have: zero order momentum, 1 order momentum and 2 order momentum identical with those of the elementary body, which means that it has the same inertia properties as the elementary body.

The aim of any reduction is to make easier the mathematical calculation necessary for the determination of the inertia properties, used in the study of body dynamics.

The presented method allows rapid calculation of inertial characteristics of a straight homogeneous bar.

The method that can be easily implemented in automatic calculation programs represents a valuable instrument in the activity of machine and mechanical tools design. The two selected applications confirm its advantages.

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