GROUPOID MORPHISMS IN TERMS OF ACTIONS

Assoc. Prof. PhD. Mădălina Roxana BUNECI, University Constantin Brâncuşi of Târgu-Jiu, ROMÂNIA

ABSTRACT. *The purpose of this paper is to introduce a notion of morphism between two groupoid defined in terms of groupoid actions and we prove that groupoids as objects and equivalence classes of certain type of such morhisms form category which is equivalent to the category introduced in [2].*

KEYWORDS: groupoid; action; morphism; category.

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1. INTRODUCTION

A symmetry is a transformation of a "space"/"object" that preserves its "structure". Groups are mathematical representations of global symmetries while groupoids are a mathematical representations of local symmetries (symmetries of "parts"). Each element of the group/groupoid corresponds to a way in which the underlying "object"/"part" is symmetric. The action of a group/groupoid on a set details the way the set transforms under the symmetry described by the group/groupoid.

Category theory provides a common language for many areas of mathematics. A category consists of a class of objects and a class of arrows (or morphisms) between the objects satisfying associativity property for the composition of arrows and the existence of identity arrows (see [5] for exact definition).

In this paper we introduce a notion of morphism between two groupoid defined in terms of groupoid actions and we prove that groupoids as objects and equivalence classes in a suitable sense of certain type of such morhisms form category which is equivalent to the category introduced in [2]. Thus this notion of mophism is a generalization of those in [1-4] and [6-7].

We shall use the same definitions and notation as in [1-4] and [6] (except that we denote the source/domain map of a groupoid with d instead s). The difference with the definition of action used in this paper and the definition in [6] is that we do not assume that the momentum map is surjective and open.

2. MORPHISMS OF GROUPOIDS IN TERMS OF ACTIONS

Definition 2.1. Let Γ and G be two groupoids. By an algebraic morphism from Γ to G we mean a triple consisting in a right action of Γ on X, a right action of G on Y together with a (groupoid) homomorphism $c : X\times\Gamma \to Y\times G$. We use the notation $\Gamma \overset{X,\,c,\,Y}{- - - >} G$.

The morphism is said continuous (or topological morphism) if the actions and the homomorphism c are continuous maps (assuming that X and Y are topological spaces as well as Γ and G are topological groupoids).

Lemma 2.2 Let Γ and G be two groupoids. Let us assume that G acts to the right on a set Y . If $\beta : \Gamma \to G$ is a groupoid homomorphism and $\alpha : \Gamma^{(0)} \to Y$ is a map satisfying the following two conditions

$$\sigma(\alpha(u)) = \beta(u), \text{ for all } u \in \Gamma^{(0)}$$
$$\alpha(r(\gamma)) \cdot \beta(\gamma) = \alpha(d(\gamma)), \text{ for all } \gamma \in \Gamma,$$

then $c : \Gamma \to Y \times G$ defined by
$$c(\gamma) = (\alpha(r(\gamma)), \beta(\gamma)), \text{ for all } \gamma \in \Gamma,$$
is a groupoid homomorphism.

Proof. For all $\gamma \in \Gamma$, we have
$$\begin{aligned} d(c(\gamma)) &= d(\alpha(r(\gamma)), \beta(\gamma)) \\ &= (\alpha(r(\gamma)) \cdot \beta(\gamma), d(\beta(\gamma))) \\ &= (\alpha(d(\gamma)), \beta(d(\gamma))) \\ &= c(d(\gamma)) \end{aligned}$$
$$\begin{aligned} r(c(\gamma)) &= (\alpha(r(\gamma)), r(\beta(\gamma))) \\ &= (\alpha(r(\gamma)), \beta(r(\gamma))) \\ &= c(r(\gamma)). \end{aligned}$$
Also for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ we have
$$\begin{aligned} c(\gamma_1\gamma_2) &= (\alpha(r(\gamma_1)), \beta(\gamma_1\gamma_2)) \\ &= (\alpha(r(\gamma_1)), \beta(\gamma_1)\beta(\gamma_2)) \\ &= (\alpha(r(\gamma_1)), \beta(\gamma_1))(\alpha(r(\gamma_1)) \cdot \beta(\gamma_1), \beta(\gamma_2)) \\ &= c(\gamma_1)(\alpha(d(\gamma_1)), \beta(\gamma_2)) \\ &= c(\gamma_1)(\alpha(r(\gamma_2)), \beta(\gamma_2)) \\ &= c(\gamma_1)c(\gamma_2). \end{aligned}$$

Lemma 2.3. Let G be a groupoid acting on a set Y . If Γ is a groupoid and $c : \Gamma \to Y \times G$ is a homomorphism, then there are two maps $\alpha : \Gamma^{(0)} \to Y$ and $\beta : \Gamma \to G$ such that

1. $c(\gamma) = (\alpha(r(\gamma)), \beta(\gamma))$ for all $\gamma \in \Gamma$.
2. $\beta : \Gamma \to G$ is a groupoid homomorphism
3. $\alpha : \Gamma^{(0)} \to Y$ is a map satisfying the following two conditions

$$\sigma(\alpha(u)) = \beta(u), \text{ for all } u \in \Gamma^{(0)}$$
$$\alpha(r(\gamma)) \cdot \beta(\gamma) = \alpha(d(\gamma)), \text{ for all } \gamma \in \Gamma.$$
If c is continuous (assuming Γ and G are topological groupoids, Y is topological space and the action of G on Y is continuous), then α and β are continuous maps.

Proof For every $\gamma \in \Gamma$, let $\beta(\gamma) = pr_2(c(\gamma))$ and $\tilde{\alpha}(\gamma) = pr_1(c(\gamma))$. Since $r(c(\gamma)) = c(r(\gamma))$, it follows that
$$(\tilde{\alpha}(\gamma), r(\beta(\gamma))) = (\tilde{\alpha}(r(\gamma)), \beta(r(\gamma))).$$

Thus $\tilde{\alpha}(\gamma)=\tilde{\alpha}(r(\gamma))$ and $r(\beta(\gamma))=\beta(r(\gamma))$ for all $\gamma\in\Gamma$. Let $\alpha : \Gamma^{(0)}\to Y$ be defined by

$$\alpha(u)=\tilde{\alpha}(u)=pr_1(c(u)) \text{ for all } u\in\Gamma^{(0)}.$$

We have $\alpha(r(\gamma))=\tilde{\alpha}(r(\gamma))=\tilde{\alpha}(\gamma)$ for all $\gamma\in\Gamma$, and therefore

$$c(\gamma)=(\alpha(r(\gamma)),\beta(\gamma)) \text{ for all } \gamma\in\Gamma.$$

The fact that $c(\alpha(u),\beta(u))=c(u)\in Y\times G$, implies that $\sigma(\alpha(u))=\beta(u)$, for all $u\in\Gamma^{(0)}$. Since $d(c(\gamma))=c(d(\gamma))$, it follows that

$$(\alpha(r(\gamma))\cdot\beta(\gamma),d(\beta(\gamma)))=(\alpha(d(\gamma)),\beta(d(\gamma))).$$

Hence $\alpha(r(\gamma))\cdot\beta(\gamma)=\alpha(d(\gamma))$ and $d(\beta(\gamma))=\beta(d(\gamma))$ for all ☹🗄 ✍. For $(\gamma_1,\gamma_2)\in\Gamma^{(2)}$ using the fact that $c(\gamma_1\gamma_2)=c(\gamma_1)c(\gamma_2)$, we obtain

$$\begin{aligned}(\alpha(r(\gamma_1)),\beta(\gamma_1\gamma_2))&=(\alpha(r(\gamma_1)),\beta(\gamma_1))(\alpha(r(\gamma_2)),\beta(\gamma_2))\\&=(\alpha(r(\gamma_1)),\beta(\gamma_1)\beta(\gamma_2)),\end{aligned}$$

and consequently, $\beta(\gamma_1\gamma_2)=\beta(\gamma_1)\beta(\gamma_2)$.

Notation 2.4. For an algebraic (respectively, a topological) morphism $\Gamma \xrightarrow{X,\,c,\,Y} G$ we denote the applications α, respectively β from the preceding lemma by $c^{(s)} : X\to Y$ (we identify the unit space of $X\times\Gamma$ with X and the unit space of $Y\times G$ with Y), respectively $c^{(g)} : X\times\Gamma\to G$. Thus for all $(x,\gamma)\in X\times\Gamma$ we have

$$c(x,\gamma)=(c^{(s)}(x),c^{(g)}(x,\gamma)),$$

where

1. $c^{(g)} : X\times\Gamma\to G$ is a groupoid homomorphism
2. $c^{(s)} : X\to Y$ is a map satisfying the following two conditions

$$\sigma(c^{(s)}(x))=c^{(g)}(x,\sigma(x)), \text{ for all } x\in X$$

$$c^{(s)}(x)\cdot c^{(g)}(x,\gamma)=c^{(s)}(x\cdot\gamma), \text{ for all } (x,\gamma)\in X\times\Gamma.$$

Definition 2.5. Two algebraic (respectively, topological) morphisms $\Gamma \xrightarrow{X_1,\,c_1,\,Y_1} G$ and $\Gamma \xrightarrow{X_2,\,c_2,\,Y_2} G$ are called equivalent if there are two bijective maps (respectively, homeomorphisms) $\phi : X_1\to X_2$ and $\psi : Y_1\to Y_2$ such that

1. $\phi : X_1\to X_2$ is Γ - equivariant
2. $\psi : Y_1\to Y_2$ is G - equivariant
3. $\psi(c_1^{(s)}(x))= c_2^{(s)}(\phi(x))$
4. $c_1^{(g)}(x,\gamma)=c_2^{(g)}(\phi(x),\gamma)$

The notion of equivalent morphism defined above is an equivalence relation.

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3. COMPOSITION OF MORPHISMS

Lemma 3. 1. Let $\Gamma_1 \overset{X_1,\, c_1,\, Y_1}{--\!\!\!-\!\!>} \Gamma_2$ and $\Gamma_2 \overset{X_2,\, c_2,\, Y_2}{---\!\!>} \Gamma_3$ be two algebraic morphisms. Let us define

$$X_{21} = \left\{ (x_2, x_1) \in X_2 \times X_1 \ : \ \sigma(x_2) = \sigma\left(c_1^{(s)}(x_1)\right) = \tilde{c}_1\left(x_1, \sigma(x_1)\right) \right\}$$

Then

1. $\left((x_2, x_1), \gamma\right) \mapsto \left(x_2 \cdot c_1^{(g)}(x_1, \gamma), x_1 \cdot \gamma\right)$ defines a right action of Γ_1 on X_{21} with momentum map $(x_2, x_1) \mapsto \sigma(x_1)$.
2. $\left((x_2, x_1), \gamma\right) \mapsto c_2\left(x_2, c_1^{(g)}(x_1, \gamma)\right) = \left(c_2^{(s)}(x_2), c_2^{(g)}\left(x_2, c_1^{(g)}(x_1, \gamma)\right)\right)$ defines a homomorphism c_{21} from $X_{21} \times \Gamma_1$ to $Y_2 \times \Gamma_3$.

Therefore $\Gamma_1 \overset{X_{21},\, c_{21},\, Y_3}{---\!\!>} \Gamma_3$ is an algebraic morphism. If $\Gamma_1 \overset{X_1,\, c_1,\, Y_1}{---\!\!>} \Gamma_2$ and $\Gamma_2 \overset{X_2,\, c_2,\, Y_2}{---\!\!>} \Gamma_3$ are topological morphism, then $\Gamma_1 \overset{X_{21},\, c_{21},\, Y_3}{---\!\!>} \Gamma_3$ is a topological morphism.

Proof. Let us check that $\left((x_2, x_1), \gamma\right) \mapsto \left(x_2 \cdot c_1^{(g)}(x_1, \gamma), x_1 \cdot \gamma\right)$ defines an action. We have

$$\begin{aligned} \sigma\left((x_2, x_1) \cdot \gamma\right) &= \sigma\left(x_2 \cdot c_1^{(g)}(x_1, \gamma), x_1 \cdot \gamma\right) \\ &= \sigma\left(x_1 \cdot \gamma\right) = d(\gamma) \end{aligned}$$

and

$$\begin{aligned} (x_2, x_1) \cdot \sigma(x_1) &= x_2 \cdot c_1^{(g)}\left(x_1, \sigma(x_1)\right), x_1 \cdot \sigma(x_1) \\ &= (x_2, x_1). \end{aligned}$$

If $(\gamma_1, \gamma_2) \in \Gamma_1^{(2)}$, then

$$\begin{aligned} \left((x_2, x_1) \cdot \gamma_1\right) \cdot \gamma_2 &= \sigma\left(x_2 \cdot c_1^{(g)}(x_1, \gamma_1), x_1 \cdot \gamma_1\right) \cdot \gamma_2 \\ &= \sigma\left(x_2 \cdot c_1^{(g)}(x_1, \gamma_1) \cdot c_1^{(g)}(x_1 \cdot \gamma_1, \gamma_2), (x_1 \cdot \gamma_1) \cdot \gamma_2\right) \\ &= \sigma\left(x_2 \cdot c_1^{(g)}(x_1, \gamma_1\gamma_2), x_1 \cdot (\gamma_1\gamma_2)\right) \\ &= (x_2, x_1) \cdot (\gamma_1\gamma_2). \end{aligned}$$

Let us check that c_{21} is a homomorphism. We have

$$\begin{aligned} d\left(c_{21}\left((x_2, x_1), \gamma\right)\right) &= d\left(c_2\left(x_2, c_1^{(g)}(x_1, \gamma)\right)\right) \\ &= c_2\left(x_2 \cdot c_1^{(g)}(x_1, \gamma), c_1^{(g)}\left(x_1 \cdot \gamma, d(\gamma)\right)\right) \\ &= c_{21}\left(\left(x_2 \cdot c_1^{(g)}(x_1, \gamma), x_1 \cdot \gamma\right), d(\gamma)\right) \\ &= c_{21}\left((x_2, x_1) \cdot \gamma, d(y)\right) \\ &= c_{21}\left(d\left(\left((x_2, x_1), \gamma\right)\right)\right) \end{aligned}$$

$$r\left(c_{21}\left(\left(x_2,x_1\right),\gamma\right)\right)=r\left(c_2\left(x_2,c_1^{(g)}\left(x_1,\gamma\right)\right)\right)$$
$$=c_2\left(x_2,c_1^{(g)}\left(x_1,r\left(\gamma\right)\right)\right)$$
$$=c_{21}\left(\left(x_2,x_1\right),r\left(\gamma\right)\right)$$
$$=c_{21}\left(r\left(\left(x_2,x_1\right),\gamma\right)\right)$$
$$c_{21}\left(\left(x_2,x_1\right),\gamma_1\gamma_2\right)=c_2\left(x_2,c_1^{(g)}\left(x_1,\gamma_1\gamma_2\right)\right)$$
$$=c_2\left(x_2,c_{11}^{(g)}\left(x_1,\gamma_1\right)c_1^{(g)}\left(x_1\cdot\gamma_1,\gamma_2\right)\right)$$
$$=c_2\left(x_2,c_1^{(g)}\left(x_1,\gamma_1\right)\right)c_2\left(x_2\cdot c_1^{(g)}\left(x_1,\gamma_1\right),c_1^{(g)}\left(x_1\cdot\gamma_1,\gamma_2\right)\right)$$
$$=c_{21}\left(\left(x_2,x_1\right),\gamma_1\right)c_{21}\left(\left(x_2,x_1\right)\cdot\gamma_1,\gamma_2\right).$$

Definition 3.2. Let $\Gamma_1 \overset{X_1,\,c_1,\,Y_1}{--->} \Gamma_2$ and $\Gamma_2 \overset{X_2,\,c_2,\,Y_2}{--->} \Gamma_3$ be two algebraic (respectively, topological) morphisms. Then we shall call the morphism $\Gamma_1 \overset{X_{21},\,c_{21},\,Y_{21}}{--->} \Gamma_3$ defined in the Lemma 3.1 the composition of $\Gamma_2 \overset{X_2,\,c_2,\,Y_2}{--->} \Gamma_3$ and $\Gamma_1 \overset{X_1,\,c_1,\,Y_1}{--->} \Gamma_2$. We write $X_{21}=X_2*X_1$, Y_{21} = Y_2 and $c_{21}=c_2c_1$.

Lemma 3.3. The composition of morphisms defined above is compatible with the classes of equivalent morphisms.

Proof. Let $\Gamma_1 \overset{X_1,\,c_1,\,Y_1}{--->} \Gamma_2$ and $\Gamma_1 \overset{X_2,\,c_2,\,Y_2}{--->} \Gamma_2$ be two equivalent morphisms and let $\phi_1 : X_1 \to X_2$ and $\psi_1 : Y_1 \to Y_2$ be two bijective (respectively, homeomorphic) maps such that

$\phi_1 : X_1 \to X_2$ is Γ -equivariant

$\psi_1 : Y_1 \to Y_2$ is G -equivariant

$\psi_1\left(c_1^{(s)}\left(x\right)\right)= \quad c_2^{(s)}\left(\phi_1\left(x\right)\right)$

$c_1^{(g)}\left(x,\gamma\right)=c_2^{(g)}\left(\phi_1\left(x\right),\gamma\right)$

Let $\Gamma_2 \overset{X_3,\,c_3,\,Y_3}{--->} \Gamma_3$ and $\Gamma_2 \overset{X_4,\,c_4,\,Y_4}{--->} \Gamma_3$ be two equivalent morphisms and let $\phi_2 : X_3 \to X_4$ and $\psi_2 : Y_3 \to Y_4$ be two bijective (respectively, homeomorphic) maps such that

$\phi_2 : X_3 \to X_4$ is Γ -equivariant

$\psi_2 : Y_3 \to Y_4$ is G -equivariant

$\psi_2\left(c_3^{(s)}\left(x\right)\right)= \quad c_4^{(s)}\left(\phi_2\left(x\right)\right)$

$c_3^{(g)}\left(x,\gamma\right)=c_4^{(g)}\left(\phi_2\left(x\right),\gamma\right)$ for all $\left(x,\gamma\right)\in X_1\times\Gamma$.

Let us denote

$$X_3*X_1\overset{not}{=}X_{31}=\left\{\left(x_3,x_1\right)\in X_3\times X_1\ :\ \sigma\left(x_3\right)=\sigma\left(c_1^{(s)}\left(x_1\right)\right)\right\}$$

$$X_4*X_2\overset{not}{=}X_{42}=\left\{\left(x_4,x_2\right)\in X_4\times X_3\ :\ \sigma\left(x_4\right)=\sigma\left(c_2^{(s)}\left(x_3\right)\right)\right\}$$

$$\phi_{21}:X_{31}\to X_{42},\ \phi_{21}\left(x_3,x_1\right)=\left(\phi_2\left(x_3\right),\phi_1\left(x_1\right)\right)$$

$$\psi_{21}:Y_{31}\to Y_{42},\ \psi_{21}=\psi_2$$

Since

$$\sigma\left(\phi_2\left(x_3\right)\right)=\sigma\left(x_3\right)=\sigma\left(c_1^{(s)}\left(x_1\right)\right)$$
$$=c_1^{(g)}\left(x_1,\sigma\left(x_1\right)\right)$$
$$=c_2^{(g)}\left(\phi_1\left(x_1\right),\sigma\left(x_1\right)\right)$$
$$=c_2^{(g)}\left(\phi_1\left(x_1\right),\sigma\left(\phi_1\left(x_1\right)\right)\right)$$
$$=\sigma\left(c_2^{(s)}\left(\phi_1\left(x_1\right)\right)\right),$$

it follows that ϕ_{21} is correctly defined. For each $\gamma\in\Gamma_1$ and $\left(x_3,x_1\right)\in X_{21}$, we have

$$\phi_{21}\left(\left(x_3,x_1\right)\cdot\gamma\right)=\phi_{21}\left(x_3\cdot c_1^{(g)}\left(x_1,\gamma\right),x_1\cdot\gamma\right)$$
$$=\left(\phi_2\left(x_3\cdot c_1^{(g)}\left(x_1,\gamma\right)\right),\phi_1\left(x_1\cdot\gamma\right)\right)$$
$$=\left(\phi_2\left(x_3\right)\cdot c_1^{(g)}\left(x_1,\gamma\right),\phi_1\left(x_1\right)\cdot\gamma\right)$$
$$=\left(\phi_2\left(x_3\right)\cdot c_2^{(g)}\left(\phi_1\left(x_1\right),\gamma\right),\phi_1\left(x_1\right)\cdot\gamma\right)$$
$$=\left(\phi_2\left(x_3\right),\phi_1\left(x_1\right)\right)\cdot\gamma$$
$$=\phi_{21}\left(x_3,x_1\right)\cdot\gamma.$$

If we denote $c_{31}=c_3c_1$ and $c_{42}=c_4c_2$, then

$$c_{31}\left(\left(x_3,x_1\right),\gamma\right)=\left(c_3^{(s)}\left(x_3\right),c_3^{(g)}\left(x_3,c_1^{(g)}\left(x_1,\gamma\right)\right)\right)$$
$$c_{42}\left(\left(x_4,x_2\right),\gamma\right)=\left(c_4^{(s)}\left(x_4\right),c_4^{(g)}\left(x_4,c_2^{(g)}\left(x_2,\gamma\right)\right)\right)$$

Consequently,

$$c_{42}^{(g)}\left(\phi_{21}\left(x_3,x_1\right)\right)=c_{42}^{(g)}\left(\left(\phi_2\left(x_3\right),\phi_1\left(x_1\right)\right),\gamma\right)$$
$$=c_4^{(g)}\left(\phi_2\left(x_3\right),c_2^{(g)}\left(\phi_1\left(x_1\right),\gamma\right)\right)$$
$$=c_4^{(g)}\left(\phi_2\left(x_3\right),c_1^{(g)}\left(x_1,\gamma\right)\right)$$
$$=c_3^{(g)}\left(x_3,c_1^{(g)}\left(x_1,\gamma\right)\right)$$

and

$$c_{42}^{(s)}\left(\phi_{21}\left(x_3,x_1\right)\right)=c_{42}^{(s)}\left(\phi_2\left(x_3\right),\phi_1\left(x_1\right)\right)$$
$$=c_4^{(s)}\left(\phi_2\left(x_3\right)\right)$$
$$=\psi_2\left(c_3^{(s)}\left(x_3\right)\right)$$
$$=\psi_{21}\left(c_{31}^{(s)}\left(x_3,x_1\right)\right)$$

It follows that $\Gamma_1\overset{X_{31},\,c_{31},\,Y_2}{--->}\Gamma_3$ and $\Gamma_2\overset{X_2,\,c_2,\,Y_2}{--->}\Gamma_4$ are equivalent algebraic (respectively, topological) morphisms.

Lemma 3.4. The composition of morphisms in the sense of Definition 3.2 is associative.

Proof. Let $\Gamma_1\overset{X_1,\,c_1,\,Y_1}{--->}\Gamma_2$, $\Gamma_2\overset{X_2,\,c_2,\,Y_2}{--->}\Gamma_3$ and $\Gamma_3\overset{X_3,\,c_3,\,Y_3}{--->}\Gamma_4$ be three algebraic (respectively, topological) morphisms. Let us denote

$$\left(X_3 * X_2\right) * X_1 \overset{not}{=} X_{(32)1}$$

$$X_3 * \left(X_2 * X_1\right) \overset{not}{=} X_{3(21)}$$

$$\left(c_3 c_2\right) c_1 \overset{not}{=} c_{(32)1}, c_3 c_2 \overset{not}{=} c_{32}$$

$$c_3\left(c_2 c_1\right) \overset{not}{=} c_{3(21)}, c_2 c_1 \overset{not}{=} c_{21}$$

Then

$$X_{(32)1} = \left\{\left(\left(x_3, x_2\right), x_1\right) : \sigma\left(x_3, x_2\right) = \sigma\left(c_1^{(s)}\left(x_1\right)\right),\ \sigma\left(x_3\right) = \sigma\left(c_2^{(s)}\left(x_2\right)\right)\right\}$$
$$= \left\{\left(\left(x_3, x_2\right), x_1\right) : \sigma\left(x_2\right) = \sigma\left(c_1^{(s)}\left(x_1\right)\right),\ \sigma\left(x_3\right) = \sigma\left(c_2^{(s)}\left(x_2\right)\right)\right\}$$

$$X_{3(21)} = \left\{\left(x_3, \left(x_2, x_1\right)\right) : \sigma\left(x_3\right) = \sigma\left(c_{21}^{(s)}\left(x_{2,} x_1\right)\right),\ \sigma\left(x_2\right) = \sigma\left(c_1^{(s)}\left(x\right)\right)\right\}$$
$$= \left\{\left(x_3, \left(x_2, x_1\right)\right) : \sigma\left(x_3\right) = \sigma\left(c_2^{(s)}\left(x_2\right)\right),\ \sigma\left(x_2\right) = \sigma\left(c_1^{(s)}\left(x\right)\right)\right\}$$

Ley $\phi : X_{(32)1} \to X_{3(21)}$ be defined by $\phi\left(\left(\left(x_3, x_2\right), x_1\right)\right) = \left(x_3, \left(x_2, x_1\right)\right)$ for all $\left(\left(x_3, x_2\right), x_1\right) \in X_{(32)1}$ and let $\psi : Y_3 \to Y_3$ be the identity function. We have

$$c_{(32)1}^{(g)}\left(\left(\left(x_3, x_2\right), x_1\right), \gamma\right) = c_{32}^{(g)}\left(\left(x_3, x_2\right), c_1^{(g)}\left(x_1, \gamma\right)\right)$$
$$= c_3^{(g)}\left(x_3, c_2^{(g)}\left(x_2, c_1^{(g)}\left(x_1, \gamma\right)\right)\right)$$
$$= c_3^{(g)}\left(x_3, c_{21}^{(g)}\left(\left(x_2, x_1\right), \gamma\right)\right)$$
$$= c_{3(21)}^{(g)}\left(\left(x_3, \left(x_2, x_1\right)\right), \gamma\right)$$
$$= c_{3(21)}^{(g)}\left(\phi\left(\left(\left(x_3, x_2\right), x_1\right)\right), \gamma\right).$$

and

$$\psi\left(c_{(32)1}^{(s)}\left(\left(x_3, x_2\right), x_1\right)\right) = c_{(32)}^{(s)}\left(x_3, x_2\right)$$
$$= c_3^{(s)}\left(x_3\right)$$
$$= c_{3(21)}^{(s)}\left(x_3, \left(x_2, x_1\right)\right)$$
$$= c_{3(21)}^{(s)}\left(\phi\left(\left(\left(x_3, x_2\right), x_1\right)\right)\right).$$

Examples 3. 5.

1. Let Γ be a groupoid and let us consider the right action of Γ on $\Gamma^{(0)}$ defined by

$$\left(u, \gamma\right) \mapsto d\left(\gamma\right),\ u = r\left(\gamma\right)$$

and the momentum map $\sigma : \Gamma^{(0)} \to \Gamma^{(0)},\ \sigma\left(u\right) = u$. Then the map $1_{\Gamma^{(0)} \times \Gamma}$ from

$$\Gamma^{(0)} \times \Gamma = \left\{\left(u, \gamma\right) \in \Gamma^{(0)} \times \Gamma : u = r\left(\gamma\right)\right\}$$

to $\Gamma^{(0)} \times \Gamma$ defined by $1_{\Gamma^{(0)} \times \Gamma}\left(u, \gamma\right) = \left(u, \gamma\right)$ is a homomorphism. Thus $\Gamma \overset{\Gamma^{(0)},\ 1_{\Gamma^{(0)}\times\Gamma},\ \Gamma^{(0)}}{- - - - >} \Gamma$ is an algebraic morphism (respectively, a topological morphism if Γ is a topological groupoid), which will be called the unit morphism of Γ and will be denoted U_Γ.

2. Each algebraic (respectively, topological) morphism $\Gamma \overset{X,\,c}{--->} G$ in the sense of [2] can be viewed as morphism $\Gamma \overset{X,\,\tilde{c},\,G^{(0)}}{--->} G$ in the sense of Definition 2.1, where $G^{(0)}$ is considered a right G-space under the action $u \cdot g = d(g)$ and $\tilde{c}(x,\gamma) = \left(c(x,\sigma(x)), c(x,\gamma)\right)$.

4. A CATEGORY OF GROUPOIDS

Proposition 4.1.

1. The class of groupoids as objects and the classes of equivalent algebraic morphisms $\Gamma \overset{X,\,c,\,Y}{--->} G$, with the property that the action of G on Y has bijective momentum map, as arrows form a category. This category is equivalent with the category G_a introduced in [2].
2. The class of topological groupoids as objects and the classes of equivalent topological morphisms $\Gamma \overset{X,\,c,\,Y}{--->} G$, with the property that momentum map of the action of G on Y is a homeomorphism, as arrows form a cateegory. This category is equivalent with the category G_t introduced in [2].

Proof. We shall check that the classes of the morphisms $\Gamma \overset{\Gamma^{(0)},\,1_{\Gamma^{(0)}?\Gamma},\,\Gamma^{(0)}}{--->} \Gamma$ (defined in Example unit) are units. Let $\Gamma \overset{X,\,c,\,Y}{--->} G$ be an algebraic (respectively, a topological) morphism. Then

$$X * \Gamma^{(0)} = \{(x,u) : \sigma(x) = u\}$$

$$c1_{\Gamma^{(0)}\times\Gamma}\left((x,u),\gamma\right) = \left(c^{(s)}(x), c^{(g)}\left(x, 1^{(g)}_{\Gamma^{(0)}\times\Gamma}(u,\gamma)\right)\right) = \left(c^{(s)}(x), c^{(g)}(x,\gamma)\right).$$

Let us define $\phi : X * \Gamma^{(0)} \to X$, $\phi(x,u) = x$ and let $\psi : Y \to Y$ be the identity map. The map ϕ is bijective (its inverse is $x \mapsto (x, \sigma(x))$) and in the topological case it is a homeomorphism. Since

$$\sigma(\phi(x,u)) = \sigma(x) = u = \sigma(x,u)$$

$$\phi((x,u)\cdot\gamma) = \phi(x\cdot\gamma, u\cdot\gamma) = x\cdot\gamma$$

$$= \phi(x,u)\cdot\gamma$$

it follows that ϕ is Γ-equivariant. Moreover, we have

$$c1_{\Gamma^{(0)}\times\Gamma}\left((x,u),\gamma\right) = \left(c^{(s)}(x), c^{(g)}(x,\gamma)\right) = c(x,\gamma) = c(\phi(x,u),\gamma).$$

Therefore $\Gamma \overset{X,\,c,\,Y}{--->} G$ and $\Gamma \overset{X*\Gamma^{(0)},\,c1_{\Gamma^{(0)}\times\Gamma},\,Y*\Gamma^{(0)}}{--->} G$ are equivalent algebraic (respectively, a topological) morphisms.

Now let $G \overset{X,\,c,\,Y}{--->} \Gamma$ be an algebraic (respectively, a topological) morphism. Then

$$\Gamma^{(0)} * X = \left\{(u,x) : u = \sigma\left(c^{(s)}(x)\right)\right\}$$

$$1_{\Gamma^{(0)}?\Gamma}c\left((u,x),\gamma\right) = \left(u, c^{(g)}(x,\gamma)\right).$$

Let us define $\phi : \Gamma^{(0)} * X \to X, \phi(u,x) = x$ and $\psi : \Gamma^{(0)} \to Y$ be the inverse of the momentum map of the action of Γ on Y. The map ϕ is bijective (its inverse is $x \mapsto \left(c^{(g)}\left(x, \sigma(x)\right), x\right)$) and in the topological case it is a homeomorphism. Since

$$\sigma(\phi(u,x)) = \sigma(x) = \sigma(x,u)$$

$$\phi((u,x)\cdot\gamma) = \phi\left(u\cdot c^{(g)}(x,\gamma), x\cdot\gamma\right)$$

$$= x\cdot\gamma = \phi(u,x)\cdot\gamma$$

it follows that ϕ is Γ-equivariant. Moreover, we have

$$1_{\Gamma^{(0)}\times\Gamma}c^{(g)}((u,x),\gamma) = c^{(g)}(x,\gamma) = c^{(g)}(\phi(u,x),\gamma)$$

$$\psi\left(1_{\Gamma^{(0)}\times\Gamma}c^{(s)}(u,x)\right) = \psi(u) = \psi\left(c^{(g)}(x,\sigma(x))\right)$$

$$= \psi\left(\sigma\left(c^{(s)}(x)\right)\right) = c^{(s)}(x)$$

$$= c^{(s)}(\phi(u,x)).$$

Therefore $G \overset{X,\ c,\ Y}{--->} \Gamma$ and $G \overset{X,\ 1_{\Gamma^{(0)}\times\Gamma}c,\Gamma^{(0)}}{--->} \Gamma$ are equivalent algebraic (respectively, a topological) morphisms.

Let us construct a functor $\Gamma\to\Gamma$, $c\to F(c)$ from the category introduced in [2] to the current category. If $\Gamma \overset{X,\ c}{--->} G$ is an algebraic (respectively, topological) morphism in the sense of [2], then let us define $F(c)$ to be the equivalence class of the algebraic (respectively, topological) morphism $\Gamma \overset{X,\ \tilde{c},\ G^{(0)}}{--->} G$, $G^{(0)}$ is considered a right G-space under the action $u\cdot g = d(g)$ and $\tilde{c}(x,\gamma) = \left(c\left(x,\sigma(x)\right), c(x,\gamma)\right)$. We can check that F is an equivalence of categories.

BIBLIOGRAPHY

[1] M. Buneci, - *Groupoid categories*, Perspectives in Operators Algebras and Mathematical Physics, 27-40, Theta Ser. Adv. Math., 8, Theta, Bucharest, 2008.

[2] M. Buneci, - *A generalization of Zakrzewski morphisms,* Advanced Studies in Contemporary Mathematics, 19 (2) (2009), 223-240.

[3] M. Buneci, - *Morphisms of discrete dynamical systems*, Discrete and Continuous Dynamical Systems, 29 (1) (2011), 91-107.

[4] M. Buneci and P. Stachura,- *Morphisms of locally compact groupoids endowed with Haar systems*, 2005, arXiv: math.OA/0511613.

[**5] T. Leinster**, - *Basic category theory*, 2016, arXiv: 1612.09375.

[6] P. Muhly, J. Reanult and D. Williams, - *Equivalence and isomorphism for groupoid C*-algebras*, J. Operator Theory **17**(1987), 3--22.

[7] S. Zakrzewski, - *Quantum and Classical pseudogroups I*, Comm. Math. Phys. 134 (1990), 347--370.

[8] S. Zakrzewski, - *Quantum and Classical pseudogroups II*, Comm. Math. Phys. 134 (1990), 371-395.