KALECKI'S MODEL OF BUSINESS CYCLE: EXISTENCE AND APPROXIMATION OF SOLUTION

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Abstract
In 1935, Michał Kalecki [1] developed the first clear and mathematically precise model of macroeconomic dynamics. Kalecki added an important real-world feature: the implementation lag (or gestation period) the time delay between deciding to invest and the new productive capacity becoming available. Our goal through current paper is to determine necessary conditions for model's solution existence and its approximation.
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1. Model formulation

A defining aspect of Kalecki's model is that productive capacities cannot be created instantly. Rather, the installation of investment goods takes a meaningful amount of time. The interval between a firm's decision to expand its capital and the moment when the new plant and equipment are fully installed and ready for production is referred to as the implementation lag or the "gestation period." For simplicity, this period is assumed to have the same duration, $\theta > 0$, for every investment project. Further let us denotes the capital stock at time t by $K(t)$ and $I(t-\theta)$ the flow of net investment decided upon at time $t-\theta$. Then we have in continuous time formulation

$$\dot{K}(t) = I(t-\theta) \tag{1}$$

The other component of investment, which is likewise subject to the same implementation lag θ, is the replacement of depreciated capital. Its level $U > 0$, is assumed to remain constant over time. Further the model assumes continuous market clearing in the short run. Ignoring inventory investment, government spending, and international trade, the remaining component of aggregate demand is consumption. For simplicity, we assume a constant aggregate saving propensity, $s \in (0,1)$. To complete the model, investment decisions are expressed as a linear function of output $Y(t)$ and capital $K(t)$ with two positive constant coefficients $a,b > 0$

$$I(t) = a \cdot Y(t) - b \cdot K(t) \tag{2}$$

where

$$Y(t) = \frac{1}{s} \cdot U + \frac{1}{s \cdot \theta}[K(t+\theta) - K(t)] \tag{3}$$

The entire model can be expressed now as a single mixed differential-difference equation in one variable (the capital stock):

$$\dot{K}(t) = \frac{a}{s} \cdot U + \frac{a}{s \cdot \theta}[K(t) - K(t-\theta)] - b \cdot K(t-\theta) \tag{4}$$

Further we remark that if the equation (4}) has at least a solution K then the equations (2) and (3) have also at least a solution and model has been completely determined.
Next, our scope is to find which conditions must be fulfilled by model parameter to have at least a continuous solution for the equation (4). For more details about Kalecki model derivation we recommend [3] and [4].

2. Model's solution existence and aproximation

Further our study will be done on the class of continuous functions $K:[-\theta,T]\to\mathbb{R}$ denoted by $C([-\theta,T],\mathbb{R})$. We consider on it, as measure of distance between two functions, the Chebisev norm defined by

$$|K|_\infty = \sup_{t\in[-\theta,T]} |K(t)|.$$

First, we will search for a solution of equation (4) which fulfills the initial condition

$$K(t) = \varphi(t), t \in [-\theta,0] \quad (5)$$

which means that we know the behavior of K during the interval $[-\theta,0]$. Then (4) + (5) are equivalent with

$$K(t) = \begin{cases} \varphi(0) + \int_0^t \left[\frac{a}{s}\cdot U + \frac{a}{s\cdot\theta}[K(u)-K(u-\theta)] - b\cdot K(u-\theta)\right] du, & t\in[0,T] \\ \varphi(t), & t\in[-\theta,0] \end{cases} \quad (6)$$

Therefore, we reduced the existence of solution for (4) +(5) to a fixed-point problem for the operator $A:C[-\theta,T]\to C[-\theta,T]$ defined by:

$$A(K)(t) = \begin{cases} \varphi(0) + \int_0^t \left[\frac{a}{s}\cdot U + \frac{a}{s\cdot\theta}[K(u)-K(u-\theta)] - b\cdot K(u-\theta)\right] du, & t\in[0,T] \\ \varphi(t), & t\in[-\theta,0] \end{cases}$$

Then for any $K_1,K_2 \in C[-\theta,T]$ we have

$$|A(K_1)(t) - A(K_2)(t)| \le \int_0^t \left[\frac{a}{s\cdot\theta}\cdot|K_1(u)-K_2(u)| + \left(\frac{a}{s\cdot\theta} - b\right)|K_1(u-\theta)-K_2(u-\theta)|\right] du \le$$

$$\le \left(2\cdot\frac{a}{s\cdot\theta} + b\right)\cdot T\cdot|K_1-K_2|_\infty.$$

From the above we get that if is known the behavior of stock function K and model parameters fulfill the condition $0 < \left(2\cdot\frac{a}{s\cdot\theta} + b\right)\cdot T < 1$ then, via the well known Banach principle, we can find a unique solution $K(\cdot,\varphi)$ of (4) + (5) as a function from $C[-\theta,T]$. Thus, we got the following result for model’s solution existence

Theorem 1. *Let us consider equations* (4) *under following conditions:*

- $\left(0 < 2\cdot\frac{a}{s\cdot\theta} + b\right)\cdot T < 1.$

Then Kalescki model (4) + (5) has a unique solution $K(\cdot,\varphi) \in C[-\theta,T]$ which can be approximated by the sequence $\{A^n(K_0)\}_{n\in\mathbb{N}}$, $K_0 \in C[-\theta,T]$ being arbitrarily chosen

3. Conclusion

Our proposal during current paper was to provide a method to find necessary conditions for the existence of solution for Kalecki’s model and to approximate and investigate its convergence. There was implemented a python script which handled the following: model’s parameters definition, implemented iteration sequence defined in Theorem 1, convergence analyzes and plot the results.

Next, we will show that when the condition from Theorem 1 is fulfilled we are able to find an approximation for the solution of Kalecki’s model. By choosing T = 1.0, theta = 0.2, N = 500, a= 1, b = 0.5, s = 10.0 and U = 1.0 we remark that Theorem 1 is satisfied and therefore there exists a solution for Kalecki's model which can be approximated as in Figure 1. On Figure 2 we can see that the error made in approximation decreases on each iteration

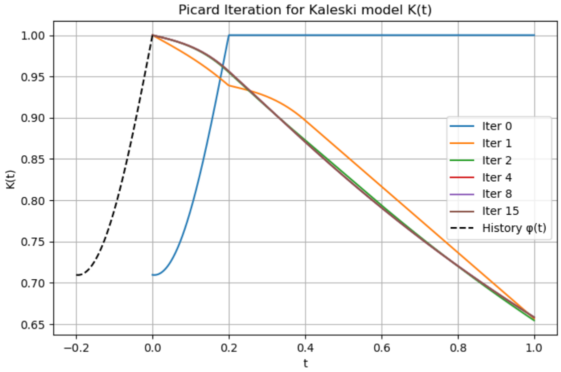

Figure 1

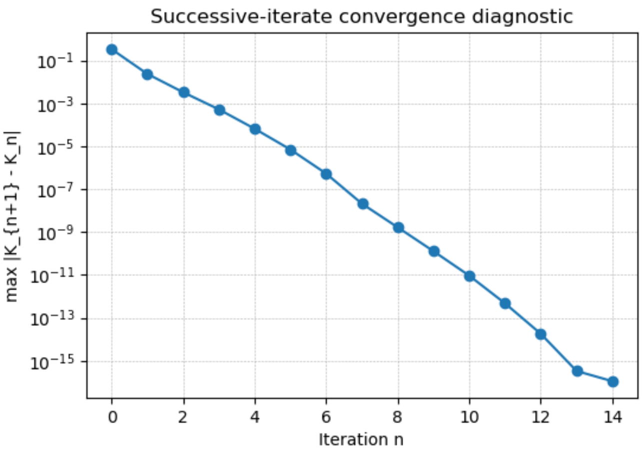

Figure 2

Next picture figure out that failing of condition from Theorem 1 leads us to a divergence of Kalescki model solution. Indeed, for T = 1.0, theta = 0.2, N = 500, a= 1, b = 0.5, s = 0.1 and U = 1.0 we get Figure 3 and Figure 4 which shows that the Kalecki's model diverges (the errors increase on each iteration). For more details about existence of solution for a Kalecki's model on class on continuous functions endowed with a Bielescki norm we recommend [2]

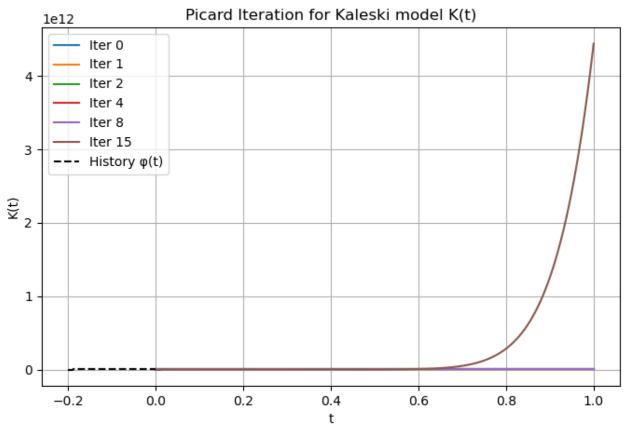

Figure 3

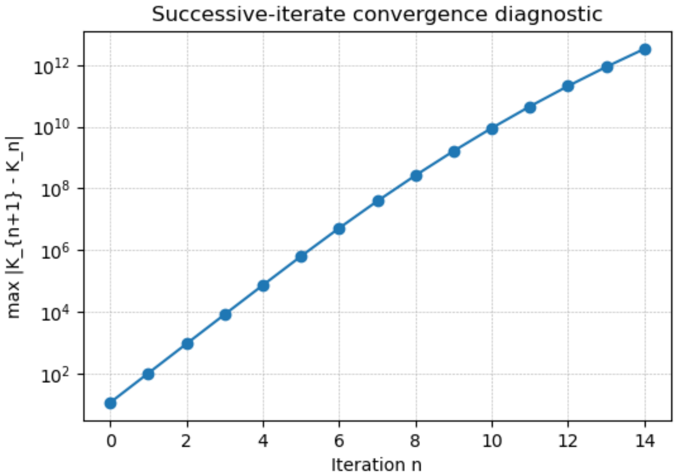

Figure 4

4. Bibliography

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