ESTIMATOR TIP CIUR PENTRU ECUAȚIILE DIFERENȚIALE PARȚIALE STOCASTICE FRACȚIONALE

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REZUMAT. *Consistența şi normalitatea asimptotică a estimatorului tip ciur al coeficientului oscilant (drift) al modelelor de ecuații diferențiale parțiale stocastice fracționale folosind un număr finit de coeficienți Fourier ai soluției sunt studiate pe baza observațiilor referitoare la un interval de timp fix [0, T]. Ecuația este condusă de un zgomot adițional care este alb în spațiu şi fracțional în timp cu parametrul Hurst H≥1/2.*

CUVINTE CHEIE: *Ecuații diferențiale parțiale stocastice, mişcarea fracțională browniană, procesul Ornstein-Uhlenbeck, memoria îndelungată, metoda de tip ciur.*

1. INTRODUCERE ŞI PRELIMINARII

Estimarea parametrilor finit dimensionali în ecuațiile diferențiale stocastice din observațiile continue şi discrete prin metodele verosimilității maxime şi Bayes sunt studiate pe larg în Bishwal (2008). Statistica ecuațiilor diferențiale parțiale stocastice (SPDEs) a fost explorată recent. În ultima vreme, procesele cu memoriei lungă şi procesele cu salturi au beneficiat de atenție în finanțe, inginerie şi fizică. Un proces cu memorie cu timp continuu îndelungat este mişcarea fracțională browniană descoperită de Kolmogorov (1940) şi ulterior studiată de Levy (1948) şi Mandelbrot şi van Ness (1968). Scopul nostru este de a studia estimația unui parametru infinit dimensional în coeficientul oscilant în SPDEs cu mişcare fracțională browniană în timp şi spațiu prin metoda sitelor.

SIEVE ESTIMATOR FOR FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. *Consistency and asymptotic normality of the sieve estimator of the drift coefficient of fractional stochastic partial differential equation models using a finite number of Fourier coefficients of the solution are studied based on observations on a fixed time interval [0, T]. The equation is driven by additive noise that is white in space and fractional in time with Hurst parameter H≥1/2.*

KEY WORDS: *Stochastic partial differential equations, fractional Brownian motion, fractional Ornstein-Uhlenbeck process, long-memory, sieve method.*

1. INTRODUCTION AND PRELIMINARIES

Finite dimensional parameter estimation in stochastic differential equations from continuous and discrete observations by maximum likelihood and Bayes methods are extensively studied in Bishwal (2008). Statistics of stochastic partial differential equations (SPDEs) has only recently been explored. Recently long memory processes and processes with jumps have received attention in finance, engineering and physics. One continuous time long memory process is the fractional Brownian motion discovered by Kolmogorov (1940) and later on studied by Levy (1948) and Mandelbrot and van Ness (1968). Our aim here is to study estimation of infinite dimensional parameter in the drift coefficient in the SPDEs with space-time fractional Brownian motion driving term by

O mişcare browniană fracțională normalizată $\{W_t^H, t \geq 0\}$ cu parametru Hurst $H \in (0,1)$ este un proces Gaussian central cu drumuri continue a cărui matrice de covarianță este dată de

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s, t \geq 0.$$

Procesul este asemănător (invariabil la scară) şi poate fi reprezentat ca o integrală stocastică referitor la mişcarea browniană standard. Pentru $H = \frac{1}{2}$, procesul este o mişcare browniană standard. Pentru $H \neq \frac{1}{2}$, fBm nu este semi-martingal şi nu este un proces Markov, ci un proces Dirichlet. Creşterile fBm sunt corelate negativ pentru $H < \frac{1}{2}$ şi corelate pozitiv pentru $H < \frac{1}{2}$ şi în acest caz demonstrează o dependență pe termen lung. Parametrul H care se mai numeşte şi parametru cu auto-similitudine, măsoară intensitatea dependenței pe termen lung. Procesul ARIMA(p,d,q) cu o parte autoregresivă de ordinul p, mişcând partea medie a ordinului q şi parametrul de diferență fracțională $d \in (0, 0.5)$ procesul converge în sens Donsker către fBm.

Lucrarea discută estimarea parametrului necunoscut $\theta(.)$ care aparține unui $\Theta \subset L_2$ deschis în ecuația de evoluție stocastică

$$dU_t(x) + (\mathbf{A}_0 + \theta\mathbf{A}_1)U_t(x)dt = dW_t^H(x), \ t \in [0,T], x \in G, \quad U_0 = 0 \ (1.1)$$

unde $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{M}_j$ sunt operatori liniari pe un domeniu mărginit din $\mathbb{R}^d$ cu ordinele m_0 şi m_1 respectiv cu $m_1 \geq m - d/2$ unde $2m = \max(m_0, m_1)$, $\{W_t^H(x)\}$ este o mişcare fracțională browniană cilindrică cu $H > 1/2$ bazată pe observarea soluției $U(t),\ t \in [0,T],\ x \in G$. În funcție de operatorii din ecuație, problema estimării poate fi regulată sau singulară. Ecuația poate fi diagnosticată dacă operatorii au un sistem comun de funcții proprii şi acest sistem are o bază ortonormală într-un spațiu Hilbert adecvat.

the method of sieves.

A normalized fractional Brownian motion $\{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a Dirichlet process. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for $H < \frac{1}{2}$ and in this case they display long-range dependence. The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence. The ARIMA(p,d,q) process with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0, 0.5)$ process converge in Donsker sense to fBm.

The paper considers estimation of the unknown parameter $\theta(.)$ belonging to an open sent $\Theta \subset L_2$ in the stochastic evolution equation

where $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{M}_j$ are linear operators on a smooth bounded domain $\mathbb{R}^d$ with orders m_0 and m_1 respectively with $m_1 \geq m - d/2$ where $2m = \max(m_0, m_1)$, $\{W_t^H(x)\}$ is a cylindrical fractional Brownian motion with $H > 1/2$ based on the observations of the solution $U(t),\ t \in [0,T],\ x \in G$. Depending on the operators in the equation the estimation problem can be regular or singular. The equations is diagonalizable if the operators have a common system of eigenfunctions and this system has a orthonormal basis in a

Mişcarea fracțională browniană în timp şi spațiu are extensia $W_t^H = \sum_{j\geq 1} {}^j W_t^H h_j$ unde ${}^j W_t^H, j \geq 1$ sunt fBm-uri independente şi h_j este o bază ortonormală în G. Următoarele extensii rezultă $U_t(x) = \sum_{j\geq 1} {}^j U_t \, h_j$ unde

suitable Hilbert space.
The space-time fractional Brownian motion has the expansion $W_t^H = \sum_{j\geq 1} {}^j W_t^H h_j$ where ${}^j W_t^H, j \geq 1$ are independent fBm's and h_j is an othonormal basis in G. The following expansion holds $U_t(x) = \sum_{j\geq 1} {}^j U_t \, h_j$ where

$$ {}^j U_t = ({}^0 U, h_j)_0 \, e^{-(\theta \nu_j + \rho_j)t} + \int_0^t e^{-(\theta\nu_j + \rho_j)(t-s)} d\, {}^j W_s^H $$

sunt procesele fracționale independente Ornstein-Uhlenbeck cu o dimensiune. Fie ${}^1U_t, {}^2U_t, \cdots, {}^nU_t$ datele observate pentru $t \in [0,T]$.

Fie $\mu_j(\theta) := \theta\nu_j + \rho_j$ unde ρ_j şi ν_j sunt proprii-valori ale lui $\mathbf{A}_0$ şi respectiv $\mathbf{A}_1$.

Să presupunem $\mu_j(\theta) > 0$ pentru toate j suficient de mari.

Fiecare ${}^jU_t$ este un proces fracțional Ornstein-Uhlenbeck care satisface

$$ d\,{}^jU_t = -\mu_j(\theta)\, {}^jU_t dt + {}^j W_t^H, \quad {}^jU_0 = 0. $$

Datorită independenței ${}^jW_t^H$ pentru diverse j, procesele ${}^1U, {}^2U, \cdots, {}^nU$ sunt independente. MLE este dată de

are independent one dimensional fractional Ornstein-Uhlenbeck processes. Let the data ${}^1U_t, {}^2U_t, \cdots, {}^nU_t$ be observed for $t \in [0,T]$.

Let $\mu_j(\theta) := \theta\nu_j + \rho_j$ where ρ_j and ν_j are eigenvalues of $\mathbf{A}_0$ and $\mathbf{A}_1$ respectively.

We assume $\mu_j(\theta) > 0$ for all sufficiently large j.

Each ${}^jU_t$ is a fractional Ornstein-Uhlenbeck process satisfying

$$ d\,{}^jU_t = -\mu_j(\theta)\, {}^jU_t dt + {}^j W_t^H, \quad {}^jU_0 = 0. $$

Because of independence of ${}^jW_t^H$ for different j, the processes ${}^1U, {}^2U, \cdots, {}^nU$ are independent.

The MLE is given by

$$ \hat{\theta}_T = -\frac{\sum_{j=1}^n \int_0^T \nu_j \; {}^jQ_s (d_j Z_s + \rho_j {}^jQ_s dv_s^H)}{\sum_{j=1}^n \int_0^T \nu_j^2 \; {}^jQ_s^2 dv_s^H}. $$

Să presupunem că există un număr real pozitiv γ astfel încât $\sum_{j\geq 1}(1 + |\mu_j(\theta)|)^{-\gamma} < \infty$. Să definim

Assume that there exists a positive real number γ such that $\sum_{j\geq 1}(1 + |\mu_j(\theta)|)^{-\gamma} < \infty$. Define

$$ \kappa_H := 2H\Gamma(3/2 - H)\Gamma(H + 1/2), \quad k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H}, \quad \lambda_H := \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2 - H)}, $$

$$ v_t \equiv v_t^H := \lambda_H^{-1} t^{2-2H}, \quad {}^jM_t^H := \int_0^t k_H(t,s) d\, {}^jW_s^H $$

De la Norros *et al.* (1999) se ştie că M_t^H este o martingală gaussiană, numită *martingală fundamentală* a cărei funcție de variație $\langle M^H \rangle_t$ este v_t^H. Filtrarea naturală a martingalului M^H coincide cu filtrarea naturală a fBm W^H din moment ce

From Norros *et al.* (1999) it is well known that M_t^H is a Gaussian martingale, called the *fundamental martingale* whose variance function $\langle M^H \rangle_t$ is v_t^H. The natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H

${}^jW_t^H := \int_0^t K(t,s)d\,{}^jM_s^H$ depinde de $H \in (1/2,1)$ unde

since ${}^jW_t^H := \int_0^t K(t,s)d\,{}^jM_s^H$ holds for $H \in (1/2,1)$ where

$$K_H(t,s) := H(2H-1)\int_s^t r^{h-\frac{1}{2}}(r-s)^{H-\frac{3}{2}},\ \ 0 \le s \le t$$

şi pentru $H = 1/2$ este folosită convenția $K_{1/2} \equiv 1$.

Să definim ${}^jQ_t := \frac{d}{dv_t}\int_0^t k_H(t,s)\,{}^jU_s ds.$ Este foarte uşor să observăm că

and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define ${}^jQ_t := \frac{d}{dv_t}\int_0^t k_H(t,s)\,{}^jU_s ds.$ It is easy to see that

$${}^jQ_t = \frac{\lambda_H}{2(2-2H)}\left\{t^{2H-1}\,{}^jZ_t + \int_0^t r^{2H-1}d\,{}^jZ_s\right\}.$$

U admite reprezentarea ${}^jU_t = \int_0^t K_H(t,s)d\,{}^jZ_s$. Filtrarea naturală generată de procesul semi-martingalului fundamental ${}^jZ_t = \theta\int_0^t {}^jQ_s dv_s + {}^jM_t^H$ şi de procesul ${}^jU$ coincid, vezi Kleptsyna şi Le Breton (2002). Informațiile disponibile pentru U şi Z sunt strict echivalente.

Fie ca realizarea $\{{}^jU_t, 0 \le t \le T,\ 1 \le j \le n\}$ sau echivalentul $\{{}^jZ_t, 0 \le t \le T,\ 1 \le j \le n\}$ să fie reprezentate de ${}_1^nZ_0^T$. Fie $P_\theta^{T,n}$ măsura generată în spațiul (C_T, B_T) funcțiilor continue pe $[0,T]$ cu σ -algebra B_T Borel asociată generată sub o normă supremă de ${}_1^nZ_0^T$ şi $P_0^{T,n}$ să fie măsura Wiener standard. Aplicând formula fracțională Girsanov, când θ este valoarea reală a parametrului, $P_\theta^{T,n}$ este absolut continuă referitor la P_0^T şi derivata Radon-Nikodym (probabilitate) a lui $P_\theta^{T,n}$ cu referire la $P_0^{T,n}$ pe baza ${}_1^nZ_0^T$ dată de

U admits the representation ${}^jU_t = \int_0^t K_H(t,s)d\,{}^jZ_s$. The natural filtration generated by the fundamental semimartingale process ${}^jZ_t = \theta\int_0^t {}^jQ_s dv_s + {}^jM_t^H$ and the process ${}^jU$ coincide, see Kleptsyna and Le Breton (2002). The available information for U and Z are strictly equivalent.

Let the realization $\{{}^jU_t, 0 \le t \le T,\ 1 \le j \le n\}$ or equivalently $\{{}^jZ_t, 0 \le t \le T,\ 1 \le j \le n\}$ be denoted by ${}_1^nZ_0^T$. Let $P_\theta^{T,n}$ be the measure generated on the space (C_T, B_T) of continuous functions on $[0,T]$ with the associated Borel σ -algebra B_T generated under the supremum norm by the process ${}_1^nZ_0^T$ and $P_0^{T,n}$ be the standard Wiener measure. Applying fractional Girsanov formula, when θ is the true value of the parameter, $P_\theta^{T,n}$ is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (likelihood) of $P_\theta^{T,n}$ with respect to $P_0^{T,n}$ based on ${}_1^nZ_0^T$ is given by

$$L_{T,n}(\theta) := \frac{dP_\theta^{T,n}}{dP_0^{T,n}}({}_1^nZ_0^T) = \exp\left\{-\sum_{j=1}^n \mu_j(\theta)\int_0^T {}^jQ_t d\,{}^jZ_t - \sum_{j=1}^n \frac{(\mu_j(\theta))^2}{2}\int_0^T {}^jQ_t^2 dv_t\right\}. (1.3)$$

Să considerăm funcția punctaj, derivata funcției probabilității logaritmice dată de

Consider the score function, the derivative of the log-likelihood function, which is given by

$$Y_{n,T}(\theta) := \sum_{j=1}^{n}\int_0^T \nu_j{}^jQ_t d^jZ_t - \sum_{j=1}^{n}(2\theta\nu_j^2 + 2\rho_j\nu_j)\int_0^{Tj} Q_t^2 dv_t. \ (1.4)$$

Procesul admite reprezentarea

The log-likelihood ratio process admits the representation

$$\Lambda_{T,n} := \log\frac{dP_{\theta_1}^{T,n}}{dP_{\theta_0}^{T,n}}({}_1^nZ_0^T) = -(\mu_j(\theta_1)-\mu_j(\theta_0))\sum_{j=1}^{n}\int_0^{Tj} Q_t d^jM_t - \frac{(\mu_j(\theta_1)-\mu_j(\theta_0))^2}{2}\sum_{j=1}^{n}\int_0^{Tj} Q_t^2 dv_t. \ (1.5)$$

2. ESTIMATORUL TIP CIUR

Grenander (1981) a introdus metoda ciurului pentru a estima parametrii dimensionali infiniți. Estimarea de tip ciur pentru ecuațiile diferențiale stocastice liniare este studiată de Nguyen şi Pham (1982).

Fie $(\Omega,\mathbf{F},P)$ un spațiu complet de probabilitate pe care se defineşte SPDE parabolică

$$du^\theta(t,x) = A^\theta u^\theta(t,x)dt + dW(t,x),\ 0 \le t \le T,$$

Cu condiții Dirichlet

2. SIEVE ESTIMATOR

Grenander (1981) introduced the method of sieves for estimating infinite dimensional parameters. Sieve estimation for linear stochastic differential equations is studied in Nguyen and Pham (1982).

Let $(\Omega,\mathbf{F},P)$ be a complete probability space on which is defined the parabolic SPDE

$$du^\theta(t,x) = A^\theta u^\theta(t,x)dt + dW(t,x),\ 0 \le t \le T,$$

with Dirichlet boundary conditions

$$u(0,x) = u_0(x) \ (2.2)$$

$$D^\gamma u(t,x)|_{\partial G} = 0 \ (2.3)$$

pentru orice γ cu $|\gamma| \le m-1$, unde $A^\theta = \theta A_1 + A_0$, A_1 şi A_0 sunt operatori diferențiali parțiali de ordinul m_1 şi respectiv m_2, A^θ are ordinul $2m = \max(m_1, m_0)$, $W(t,x)$ este o mişcare browniană cilindrică în $L^2([0,T]\times G)$ unde G este un domeniu mărginit din $\mathbb{R}^d$ şi $u_0 \in L_2(G)$. Aici $\theta(.) \in \Theta \subseteq L_2$ este parametrul necunoscut care trebuie estimat pe baza observațiilor câmpului $u^\theta(t,x), t\in[0,T], x\in G$. Fie θ_0 valoarea reală a parametrului necunoscut.

Aici $u^\theta(t,x)$ este observația la momentul t în punctul x . In practică, este imposibil să se observe câmpul $u^\theta(t,x)$ la toate punctele t şi x. Aşadar, se presupune că numai o proiecție dimensională finită $u^n := u^{n,\theta} = (u_1^\theta(t),\ldots,u_n^\theta(t))$, $t\in[0,T]$ a soluției ecuației (2.1) este disponibilă. Cu alte cuvinte, observăm primele cele mai mari n noduri din seria Fourier

for all indices γ with $|\gamma| \le m-1$, where $A^\theta = \theta A_1 + A_0$, A_1 and A_0 are partial differential operators of orders m_1 and m_2 respectively, A^θ has order $2m = \max(m_1, m_0)$, $W(t,x)$ is a cylindrical Brownian motion in $L^2([0,T]\times G)$ where G is a bounded domain in $\mathbb{R}^d$ and $u_0 \in L_2(G)$. Here $\theta(.) \in \Theta \subseteq L_2$ is the unknown parameter to be estimated on the basis of the observations of the field $u^\theta(t,x), t\in[0,T], x\in G$. Let θ_0 be the true value of the unknown parameter.

Here $u^\theta(t,x)$ is the observation at time t at point x . In practice, it is impossible to observe the field $u^\theta(t,x)$ at all points t and x. Hence, it is assumed that only a finite dimensional projection $u^n := u^{n,\theta} = (u_1^\theta(t),\ldots,u_n^\theta(t))$, $t\in[0,T]$ of the solution of the equation (2.1) are available. In other words, we observe the first n highest nodes in the Fourier expansion

$$u^{\theta}(t,x)=\sum_{t=1}^{\infty}u_i^{\theta}(t)\phi_i(x)$$

care corespunde unei baze ortogonale $\{\phi_i(x)\}_{i=1}^{\infty}$. Considerăm observația continuă în timp $t\in[0,T]$. Rețineți că $u_i^{\theta}(t), i\geq 1$ sunt procese Ornstein-Uhlenbeck dimensionale independente (vezi Huebner şi Rozovskii (1995)).

(H1) $m_1 \geq m-d/2$.

(H2) Operatorii A_1 şi A_0 sunt auto-conjugați din punct de vedere formal, adică pentru $i=0,1$

$$\int_G A_i uvdx=\int_G uA_i vdx \quad \text{for all } u,v\in C_0^{\infty}(G).$$

(H3) Există o vecinătate compactă Θ a θ_0 astfel încât $\{A^{\theta},\theta\in\Theta\}$ este o familie de operatori eliptici tare uniformi.

(H4) Există un sistem ortonormal complet $\{h_i\}_{i=1}^{\infty}$ in $L_2(G))$ astfel încât pentru fiecare $i=1,2,\ldots,h_i\in W_0^{m,2}(G)\cap C^{\infty}(\overline{G})$ şi $\Lambda_{\theta}h_i=\lambda_i(\theta)h_i$, şi $\mathbf{L}_{\theta}h_i=\mu_i(\theta)h_i$ pentru orice $\theta\in\Theta$ unde $\mathbf{L}_{\theta}$ este o extensie auto-conjugată închisă a A^{θ}, $\Lambda_{\theta}:=(k(\theta)I-\mathbf{L}_{\theta})^{1/2m}$, $k(\theta)$ este o constantă şi spectrul constant al operatorului Λ_{θ} este alcătuit din proprii valori $\{\lambda_i(\theta)\}_{i=1}^{\infty}$ ale multiplicităților finite şi $\mu_i=-\lambda_i^{2m}+k(\theta)$.

(H5) Operatorul A_1 este puternic eliptic uniform şi are acelaşi sistem de proprii funcții $\{h_i\}_{i=1}^{\infty}$ ca $\mathbf{L}_{\theta}$.

Pentru $\alpha>d/2$, definim spațiul Hilbert $H^{-\alpha}$ ca în Huebner and Rozovskii (1995). Fie P_{θ}^{T} măsura generată de soluția $\{u^{\theta}(t,x),t\in[0,T],x\in G\}$ la problema (2.1) - (2.3) în spațiul $\mathbf{C}([0,T];H^{-\alpha})$ cu σ -algebra $\mathbf{B}_T$ Borel asociată. Reținem că în (H1), pentru diferite θ măsurile P_{θ}^{T} sunt singulare.

Să considerăm proiecția $H^{-\alpha}$ în subspațiul $\mathbb{R}^n$. Fie $P_{\theta}^{T,n}$ măsura generată de $u^{n,\theta}$ în

corresponding to some orthogonal basis $\{\phi_i(x)\}_{i=1}^{\infty}$. We consider observation continuous in time $t\in[0,T]$. Note that $u_i^{\theta}(t), i\geq 1$ are independent dimensional Ornstein-Uhlenbeck processes (see Huebner and Rozovskii (1995)).

(H1) $m_1 \geq m-d/2$.

(H2) The operators A_1 and A_0 are formally self-adjoint, i.e., for $i=0,1$

(H3) There is a compact neighbourhood Θ of θ_0 so that $\{A^{\theta},\theta\in\Theta\}$ is a family of uniformly strongly elliptic operators.

(H4) There exists a complete orthonormal system $\{h_i\}_{i=1}^{\infty}$ in $L_2(G))$ such that for every $i=1,2,\ldots,h_i\in W_0^{m,2}(G)\cap C^{\infty}(\overline{G})$ and $\Lambda_{\theta}h_i=\lambda_i(\theta)h_i$, and $\mathbf{L}_{\theta}h_i=\mu_i(\theta)h_i$ for all $\theta\in\Theta$ where $\mathbf{L}_{\theta}$ is a closed self adjoint extension of A^{θ}, $\Lambda_{\theta}:=(k(\theta)I-\mathbf{L}_{\theta})^{1/2m},k(\theta)$ is a constant and the spectrum of the operator Λ_{θ} consists of eigen values $\{\lambda_i(\theta)\}_{i=1}^{\infty}$ of finite multiplicities and $\mu_i=-\lambda_i^{2m}+k(\theta)$.

(H5) The operator A_1 is uniformly strongly elliptic and has the same system of eigen functions $\{h_i\}_{i=1}^{\infty}$ as $\mathbf{L}_{\theta}$.

For $\alpha>d/2$, define the Hilbert space $H^{-\alpha}$ as in Huebner and Rozovskii (1995). Let P_{θ}^{T} the measure generated by the solution $\{u^{\theta}(t,x),t\in[0,T],x\in G\}$ to the problem (2.1) - (2.3) on the space $\mathbf{C}([0,T];H^{-\alpha})$ with the associated Borel σ -algebra $\mathbf{B}_T$. Note that, under (H1), for different θ the measures P_{θ}^{T} are singular.

Consider the projection of $H^{-\alpha}$ on to the subspace $\mathbb{R}^n$. Let $P_{\theta}^{T,n}$ be the measure

$\mathbf{C}[(0,T];\mathbb{R}^n)$ cu σ -algebra $\mathbf{B}_T^n$ Borel asociată.
Este un adevăr clasic (vezi Liptser-Shiryayev (1977)) ca pentru fiecare $\theta \in \Theta$, măsurile $P_\theta^{T,n}$ şi $P_{\theta_0}^{T,n}$ sunt reciproc absolut continue cu derivata Radon-Nikodym (raportul de probabilitate) date de

generated by $u^{n,\theta}$ on $\mathbf{C}[(0,T];\mathbb{R}^n)$ with the associated Borel σ -algebra $\mathbf{B}_T^n$.
It is a classical fact (see Liptser-Shiryayev (1977)) that for any $\theta \in \Theta$, the measures $P_\theta^{T,n}$ and $P_{\theta_0}^{T,n}$ are mutually absolutely continuous with Radon-Nikodym derivative (likelihood ratio) given by

$$Z_n^\theta(u) := \frac{dP_\theta^{T,n}}{dP_{\theta_0}^{T,n}}(u^n) = \exp\Big\{(\theta-\theta_0)\int_0^T (A_1 u^n(s), du^n(s))_0 - \tfrac{1}{2}(\theta^2-\theta_0^2)\int_0^T \left\|A_1 u^n(s)\right\|_0^2 ds - (\theta-\theta_0)\int_0^T (A_1 u^n(s), A_0 u^n(s))_0\, ds\Big\}. \quad (2.4)$$

Maximizând $Z_n^\theta(u)$ relativ la θ furnizează MLE dată de

Maximizing $Z_n^\theta(u)$ w.r.t. θ provides the MLE given by

$$\hat{\theta}^n := \frac{\int_0^T (A_1 u^n(s), du^n(s) - A_0 u^n(s) ds)_0}{\int_0^T \left\|A_1 u^n(s)\right\|_0^2 ds}. \quad (2.5)$$

Informațiile Fisher I_n relativ la $\frac{dP_\theta^n}{dP_{\theta_0}^n}$ sunt date de $I_n = E_{\theta_0}\int_0^T \left\|A_1 u^n(s)\right\|_0^2 ds.$
Fie V_n secvența crescătoare a subspațiilor $L^2([0,T],dt)$ cu dimensiunea finită d_n astfel încât $\bigcup_{n\ge 1} V_n$ este densă în $L^2([0,T],dt)$. Metoda sitelor (vezi Grenander (1981)) constă în maximizarea $L_n(\theta)$ pe V_n . Fie $\phi_k, k=1,2,\cdots,$ un şir de vectori independenți ai $L^2([0,T],dt)$ astfel încât $\phi_1,\cdots,\phi_{d_n}$ formează o bază a V_n pentru toate n . Atunci pentru $\theta \in V_n, \theta(\cdot) = \sum_{k=1}^{d_n}\theta_k\phi_k(\cdot)$, avem

The Fisher information I_n related to $\frac{dP_\theta^n}{dP_{\theta_0}^n}$ is given by $I_n = E_{\theta_0}\int_0^T \left\|A_1 u^n(s)\right\|_0^2 ds.$
Let V_n be increasing sequence of subspaces of $L^2([0,T],dt)$ with finite dimension d_n such that $\bigcup_{n\ge 1} V_n$ is dense in $L^2([0,T],dt)$. The method of sieve (see Grenander (1981)) consist of maximizing $L_n(\theta)$ on V_n . Let $\phi_k, k=1,2,\cdots,$ be a sequence of independent vectors of $L^2([0,T],dt)$ such that $\phi_1,\cdots,\phi_{d_n}$ form a basis of V_n for all n. Then for $\theta \in V_n, \theta(\cdot) = \sum_{k=1}^{d_n}\theta_k\phi_k(\cdot)$, we have

$$L_n(\theta) = \sum_{j=1}^{n}\int_0^T \left\{\sum_{k=1}^{d_n}\theta_k\phi_k(t)\right\}\frac{\mu_j(X(t))}{\sigma_j^2(X(t))}dX_j(t) - \frac{1}{2}\sum_{j=1}^{n}\int_0^T \left\{\sum_{k=1}^{d_n}\theta_k\phi_k(t)\right\}^2 \frac{\mu_j^2(X(t))}{\sigma_j^2(X(t))}dt.$$

Estimatorul cu site este dat de $\hat{\theta}^{(n)}(\cdot) = (A^{(n)})^{-1}B^{(n)}$ unde $\hat{\theta}^{(n)}(\cdot), B^{(n)}$ şi $(A^{(n)})$ sunt vectori şi matrice cu elemente generale $\hat{\theta}_k$, $k = 1,2,\cdots,d_n$

The sieve estimator is given by $\hat{\theta}^{(n)}(\cdot) = (A^{(n)})^{-1}B^{(n)}$ where $\hat{\theta}^{(n)}(\cdot), B^{(n)}$ and $(A^{(n)})$ are vectors and the matrix with general elements $\hat{\theta}_k$, $k = 1,2,\cdots,d_n$

$$B_k^{(n)} = \sum_{j=1}^{n}\int_0^T \phi_k(t)\}\frac{\mu_j(X(t))}{\sigma_j^2(X(t))}dX_j(t),\quad A_{ik}^{(n)} = \sum_{j=1}^{n}\int_0^T \phi_k(t)\phi_r(t)\frac{\mu_j^2(X(t))}{\sigma_j^2(X(t))}dt,\quad k,r=1,2,\cdots,d_n,$$

$$\hat{\theta}_k^{(n)} = \frac{\sum_{j=1}^{n}\int_0^T \phi_k(t)\frac{\mu_j(X(t))}{\sigma_j^2(X(t))}dX_j(t)}{\sum_{j=1}^{n}\int_0^T \phi_k(t)^2\frac{\mu_j^2(X(t))}{\sigma_j^2(X(t))}dt},\quad k=1,2,\cdots,d_n.$$

Aşadar, estimatorul ciur este $\hat{\theta}^{(n)}(\cdot)=\sum_{k=1}^{d_n}\hat{\theta}_k^{(n)}\phi_k^{(n)}(\cdot)$. Vom estima $\theta_0(t)$ într-un subspațiu Θ_n (ciur) al Θ. Estimarea este denotată de $\hat{\theta}(t)$.
Fie $h_1,h_2,\cdots$ un sistem ortonormal în Θ. Fie $\{\Theta_n, n\ge 1\}$ o secvență crescătoare a subspațiilor Θ astfel încât Θ_n este deschis de $h_1,h_2,\cdots,h_{d_n}$. Reținem că d_n este dimensiunea subspațiului Θ_n care depinde de numărul de n coeficienți Fourier.
Definim $q=2(m_1-m)/d$. Se ştie că o estimare consistentă este posibilă dacă şi numai dacă $q\ge -1$. Definim $F_{q,n}=\frac{n^{q+1}}{q+1}$ dacă $q>-1$ şi $F_{q,n}=\log n$ dacă $q=1$. Reținem că

Thus the sieve estimator is $\hat{\theta}^{(n)}(\cdot)=\sum_{k=1}^{d_n}\hat{\theta}_k^{(n)}\phi_k^{(n)}(\cdot)$. We will estimate $\theta_0(t)$ on a subspace Θ_n (sieve) of Θ. The estimate is denoted by $\hat{\theta}(t)$.
Let $h_1,h_2,\cdots$ be an orthonormal system in Θ. Let $\{\Theta_n, n\ge 1\}$ be an increasing sequence of subspaces of Θ so that Θ_n is spanned by $h_1,h_2,\cdots,h_{d_n}$. Notice that d_n is the dimension of the subspace Θ_n depends on the number n of Fourier coefficients observed.
Define $q=2(m_1-m)/d$. It is known that a consistent estimate is possible if and only if $q\ge -1$. Define $F_{q,n}=\frac{n^{q+1}}{q+1}$ if $q>-1$ and $F_{q,n}=\log n$ if $q=1$.
Note that

$$\lim_{n\to\infty}\frac{F_{q,n}}{\sum_{k=1}^{n}k^q}=1,\quad Q_{1,n}\sim\frac{\sum_{k=1}^{n}k^{4(m_1-m)/d}}{(\psi_n)^2},\quad Q_{1,n}\sim\frac{\sum_{k=1}^{n}k^{(2m_1-4m)/d}}{\psi_n}$$

$$\psi_n := \int_0^T E\left\|A_1u^n(t)\right\|_{L_2(G)}^2 dt.$$

$\widetilde{\theta}(t)=c_1\theta_0(t)$ dacă $m_0<m_1=2m,\ \widetilde{\theta}(t)=c_0$ dacă $m_1<m_0=2m$, $\widetilde{\theta}(t)=c_0+c_1\theta_0(t)$ dacă $m_0=m_1=2m.$
Fie $\theta_0^{(n)}$ proiecția ortogonală a θ_0 în θ_N. $\theta_0^{(n)}=\sum_{j=1}^{d_n}\theta_{0j}h_j(t)$. Fie

$\widetilde{\theta}(t)=c_1\theta_0(t)$ if $m_0<m_1=2m,\ \widetilde{\theta}(t)=c_0$ if $m_1<m_0=2m,\ \widetilde{\theta}(t)=c_0+c_1\theta_0(t)$ if $m_0=m_1=2m.$
Let $\theta_0^{(n)}$ be the orthogonal projection of θ_0 onto θ_N. $\theta_0^{(n)}=\sum_{j=1}^{d_n}\theta_{0j}h_j(t)$. Let

$$a_n=\left(\int_0^T h_j(s)(\nu_j\ {}^jQ_s(d^jZ_s+\rho_j{}^jQ_s dv_s^H))\right)_{j=1,2,\cdots,d_n}$$

$$J_n=\left(\int_0^T h_i(s)h_j(s)\nu_j^2\ {}^jQ_s^2 dv_s^H\right)_{i,j=1,2,\cdots,d_n}$$

Matricea J_n este reversibilă aproape sigur. Aşadar, MLE se poate scrie ca

$$\hat{\theta}^n = J_n^{-1} a_n$$

$$J_n(\hat{\theta}_0^n - \theta_0^n) = b_n + c_n$$

Propoziţia 2.1 (Consistenţa) din (A1), avem

$$P - \lim_{n\to\infty} \int_0^T |\hat{\theta}^{(n)}(t) - \theta_0(t)|^2 \, dt$$

şi $d_n \to \infty$ ca $d_n Q_{1,n} \to 0$ şi $D_n Q_{2,n} \to 0$ unde $\sup_{0\le t\le T} \sum_{j=1}^{d_n} |h_j(t)|^2 \le D_n$.

Demonstraţie. Metoda de demonstraţie este asemănătoare cu Nguyen şi Pham (1982) şi Huebner şi Lototsky (2000). Vom omite detaliile.

Propoziţia 2.2 (Normalitate asimptotică) În (A1), avem

The matrix J_n is invertible almost surely. Therefore, MLE can be written as

$$\hat{\theta}^n = J_n^{-1} a_n$$

$$J_n(\hat{\theta}_0^n - \theta_0^n) = b_n + c_n$$

Proposition 2.1 (Consistency) Under (A1), we have

$$P - \lim_{n\to\infty} \int_0^T |\hat{\theta}^{(n)}(t) - \theta_0(t)|^2 \, dt$$

and $d_n \to \infty$ as $d_n Q_{1,n} \to 0$ and $D_n Q_{2,n} \to 0$ where $\sup_{0\le t\le T} \sum_{j=1}^{d_n} |h_j(t)|^2 \le D_n$.

Proof. The method of proof is similar to Nguyen and Pham (1982) and Huebner and Lototsky (2000). We omit the details.

Proposition 2.2 (Asymptotic Normality) Under (A1), we have

$$\lim_{n\to\infty} \sqrt{F_{q,n}} \int_0^T g(t)(\hat{\theta}(t) - \theta_0(t))dt = \mathbf{N}\left(0, \int_0^T |g(t)|^2 \, \tilde{\theta}(t)dt\right)$$

pentru fiecare funcţie deterministică $g \in L_2(0,T)$ ca $d_n \to \infty$ şi $d_n^2 Q_{1,n} \to 0$.

Demonstraţie. Metoda de demonstraţie este asemănătoare cu Nguyen şi Pham (1982) şi Huebner şi Lototsky (2000). Vom omite detaliile

for every deterministic function $g \in L_2(0,T)$ as $d_n \to \infty$ and $d_n^2 Q_{1,n} \to 0$.

Proof. The method of proof is similar to Nguyen and Pham (1982) and Huebner and Lototsky (2000). We omit the details

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