SOLUŢII DE EVOLUŢIE PENTRU O CLASĂ DE ECUAŢII DIFERENŢIALE DE TIP LYAPUNOV

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REZUMAT: In acest articol este studiata problema existenţei soluţiilor de evoluţie pentru o clasă de ecuaţii diferenţiale de tip Lyapunov definite pe spaţii Banach ordonate, infinit dimensionale. Aceste ecuaţii joacă un rol important în studiul stabilităţii exponenţiale a ecuaţiilor liniare diferenţiale stochastice cu salturi Markov.

CUVINTE CHEIE: ecuaţii Lyapunov, soluţii de evoluţie, spaţii Banach ordonate, evoluţii pozitive

1. REZULTATE PRELIMINARII SI FORMULAREA PROBLEMEI

Fie *H, U* şi *V* spaţii Hilbert reale. Vom nota cu *L(H,U)* spaţiul Banach real al operatorilor liniari şi mărginiţi definiţi pe *H* cu valori în *U* şi cu E_H sub spaţiul Banach al lui *L(H) : = L(H,H)* format din toţi operatorii auto adjuncţi. Ca de obicei, vom nota cu $\langle .,. \rangle_H$ produsul scalar pe *H* şi cu $\|.\|$ normele elementelor şi operatorilor, dacă nu este specificat altfel. Un operator $A \in L(H)$ se numeşte *nenegativ* şi scriem $A \geq 0$, dacă *A* este auto-adjunct şi $\langle Ah, h \rangle_H \geq 0$ pentru toţi $h \in H$. Vom nota cu K_H conul tuturor operatorilor nenegativi definiţi pe *H* . Fie Z un interval de întregi, care poate fi finit sau infinit. Dacă *B* este un spaţiu Banach real, atunci $l_B^{\mathbf{Z}} = \{g = \{g(i)\}_{i \in \mathbf{Z}}, \|g\|_{\mathbf{Z}} = \sup_{i \in \mathbf{Z}} \|g(i)\| < \infty\}$ este un spaţiu liniar real înzestrat cu operaţiile uzuale de adunare pe componente şi înmultirea cu scalari (reali). Mai mult, $l_B^{\mathbf{Z}}$

MILD SOLUTIONS FOR A CLASS OF DIFFERENTIAL, LYAPUNOV TYPE, EQUATIONS

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ABSTRACT: We consider a class of differential equations of Lyapunov type defined on infinite dimensional ordered Banach spaces and we provide sufficient conditions for the existence of their mild solutions. These equations play an important role in the study of exponential stability of linear stochastic differential equations with Markovian jumps.

KEYWORDS: Lyapunov equations, mild solution, Banach ordered space, positive evolution

1. PRELIMINARIES AND STATEMENT OF THE PROBLEM

Let *H, U* and *V* be real separable Hilbert spaces. We will denote by *L(H,U)* the real Banach space of linear and bounded operators from *H* into *U* and by E_H the Banach subspace of *L(H):=L(H,H)* formed by all self-adjoint operators. As usual, we shall write $\langle .,. \rangle_H$ for the inner product on *H* and $\|.\|$ for norms of elements and operators, unless indicated otherwise. An operator $A \in L(H)$ is called *nonnegative* and we write $A \geq 0$, if *A* is self-adjoint and $\langle Ah, h \rangle_H \geq 0$ for all $h \in H$. We will denote by K_H the cone of all nonnegative operators from *H* . Let Z be an interval of integers, which may be finite or infinite. If *B* is a real Banach space, then $l_B^{\mathbf{Z}} = \{g = \{g(i)\}_{i \in \mathbf{Z}}, \|g\|_{\mathbf{Z}} = \sup_{i \in \mathbf{Z}} \|g(i)\| < \infty\}$ is a real linear space endowed with the usual term-wise addition and the (real) scalar multiplication. Moreover, $l_B^{\mathbf{Z}}$ is a Banach este un spatiu Banach dacă îl înzestrăm cu norma $\|.\|_{\mathbf{Z}}$. Dacă B este $L(U,H)$ sau E_H , atunci $l_B^{\mathbf{Z}}$ va fi notat $l_{L(U,H)}^{\mathbf{Z}}$ sau $l_{\mathbf{E}_H}^{\mathbf{Z}}$. Dacă $A \in l_{L(U,H)}^{\mathbf{Z}}, B \in l_{L(H,U)}^{\mathbf{Z}}$, atunci produsul AB este definit astfel $(AB)(i) = A(i)B(i), i \in \mathbf{Z}$. În mod evident $AB \in l_{L(H)}^{\mathbf{Z}}$. Vom folosi simbolul standard * pentru a nota adjunctul unui operator liniar şi mărginit. Dacă $P \in l_{L(H,U)}^{\mathbf{Z}}$, atunci $P^{[*]} \overset{\Delta}{=} \{P(i)^*, i \in \mathbf{Z}\} \in l_{L(U,H)}^{\mathbf{Z}}$. Spunem că un element $X \in l_{\mathbf{E}_H}^{\mathbf{Z}}$ este *nenegativ* (şi scriem $X \geq 0$) dacă $X(i) \geq 0$ pentru toţi $i \in Z$. Conul $\mathbf{K}_H^{\mathbf{Z}}$ format din toate elementele nenegative din $l_{\mathbf{E}_H}^{\mathbf{Z}}$ introduce ordinea următoare pe $l_{\mathbf{E}_H}^{\mathbf{Z}}$:

$X \geq Y$ dacă şi numai dacă $X - Y \in \mathbf{K}_H^{\mathbf{Z}}$.

Dacă B este un spaţiu Banach arbitrar, vom nota cu $C(J, B)$ spaţiul tuturor funcţiilor continue $G: J \to B$. Dacă H şi U sunt spaţii Hilbert separabile, folosim notaţia $C_s(J, L(H,U))$ pentru spaţiul tuturor funcţiilor tare continue $G: J \to L(H,U)$ şi $C_b(J, L(H,U))$ pentru subspaţiul lui $C_s(J, L(H,U))$ format din toate funcţiile G cu proprietatea $\sup_{t \in J}\|G(t)\| < \infty$.

Dacă $D \in l_{L(H,V)}^{\mathbf{Z}}$, $h \in l_H^{\mathbf{Z}}$ vom nota prin Dh elemental lui $l_V^{\mathbf{Z}}$ definit astfel: $Dh(i) = D(i)h(i)$, $i \in Z$. Deasemenea notam cu $C_s(J, l_{L(H,V)}^{\mathbf{Z}}))$ şi respectiv $C_b(J, l_{L(H,V)}^{\mathbf{Z}})$ mulţimile $\{G : J \to l_{L(H,V)}^{\mathbf{Z}} \mid G(.)h \in C(J, l_V^{\mathbf{Z}}), h \in l_H^{\mathbf{Z}}\}$ şi respectiv $\{G \in C_s(J, l_{L(H,V)}^{\mathbf{Z}})) \mid \sup_{t \in J}\|G(t)\|_{\mathbf{Z}} < \infty\}$.

Se observă că dacă $G \in C_s(J, l_{L(H,V)}^{\mathbf{Z}}))$ şi $F \in C_s(J, l_{L(U,H)}^{\mathbf{Z}}))$ atunci

(1) $\quad GF = \{G(t)F(t)\}_{s \in J} \in C_s(J, l_{L(U,V)}^{\mathbf{Z}}))$.

space when endowed with the norm $\|.\|_{\mathbf{Z}}$. If B is $L(U,H)$ or E_H , then $l_B^{\mathbf{Z}}$ will be denoted by $l_{L(U,H)}^{\mathbf{Z}}$ or $l_{\mathbf{E}_H}^{\mathbf{Z}}$. If $A \in l_{L(U,H)}^{\mathbf{Z}}, B \in l_{L(H,U)}^{\mathbf{Z}}$, then the product AB is defined by $(AB)(i) = A(i)B(i), i \in \mathbf{Z}$. Obviously $AB \in l_{L(H)}^{\mathbf{Z}}$. We use the standard symbol * for the adjoint of a linear and bounded operator. If $P \in l_{L(H,U)}^{\mathbf{Z}}$, then $P^{[*]} \overset{\Delta}{=} \{P(i)^*, i \in \mathbf{Z}\} \in l_{L(U,H)}^{\mathbf{Z}}$. An element $X \in l_{\mathbf{E}_H}^{\mathbf{Z}}$ is said to be *nonnegative* (we write $X \geq 0$) iff $X(i) \geq 0$ for all $i \in Z$. The cone $\mathbf{K}_H^{\mathbf{Z}}$ of all nonnegative elements of $l_{\mathbf{E}_H}^{\mathbf{Z}}$ introduces the following order on $l_{\mathbf{E}_H}^{\mathbf{Z}}$:

$X \geq Y$ iff $X - Y \in \mathbf{K}_H^{\mathbf{Z}}$.

If B is an arbitrary Banach space, we denote by $C(J, B)$ the space of all mappings $G: J \to B$ that are continuous. For any real separable Hilbert spaces H and U, we use the notation $C_s(J, L(H,U))$ for the space of all strongly continuous mappings $G: J \to L(H,U)$ and $C_b(J, L(H,U))$ for the subspace of $C_s(J, L(H,U))$ which consist of all mappings G such that

$$\sup_{t \in J}\|G(t)\| < \infty.$$

If $D \in l_{L(H,V)}^{\mathbf{Z}}$, $h \in l_H^{\mathbf{Z}}$ we denote by Dh the element of $l_V^{\mathbf{Z}}$ given by $Dh(i) = D(i)h(i)$, $i \in Z$. Also $C_s(J, l_{L(H,V)}^{\mathbf{Z}}))$ will be the set

$\{G : J \to l_{L(H,V)}^{\mathbf{Z}} \mid G(.)h \in C(J, l_V^{\mathbf{Z}}), h \in l_H^{\mathbf{Z}}\}$

and

$C_b(J, l_{L(H,V)}^{\mathbf{Z}}) = \{G \in C_s(J, l_{L(H,V)}^{\mathbf{Z}})) \mid \sup_{t \in J}\|G(t)\|_{\mathbf{Z}} < \infty\}$

. Note that if $G \in C_s(J, l_{L(H,V)}^{\mathbf{Z}}))$ and $F \in C_s(J, l_{L(U,H)}^{\mathbf{Z}}))$ then

(1) $\quad GF = \{G(t)F(t)\}_{s \in J} \in C_s(J, l_{L(U,V)}^{\mathbf{Z}}))$.

It is well known that $C_s(J, L(H,V))$ is a real Banach space when endowed with with the usual operations and the norm

Este bine cunoscut faptul că *$C_s(J, L(H,V))$* este un spaţiu Banach real înzestrat cu operaţiile uzuale şi cu norma $\|F(t)\|_T = \sup_{s\in[0,T]}\|F(s)\|$, $F \in C_s(J, L(H,V))$.

LEMA 1. *Spaţiul liniar* $C_s([0,T], l^{\mathbf{z}}_{L(H,V)}), T\in \mathbf{R}^*_+$ *este un spaţiu Banach real cu norma* $|G|_T = \sup_{s\in[0,T]}\|G(s)\|_{\mathbf{z}}$.

DEMONSTRAŢIE. Pentru orice $D \in l^{\mathbf{z}}_{L(H,V)}$, $h\in l^{\mathbf{z}}_H$, definim operatorul liniar şi mărginit $\tilde{D} \in L\left(l^{\mathbf{z}}_H, l^{\mathbf{z}}_{L(H,V)}\right)$ definit prin $\tilde{D}$ (h) = Dh. Evident

(2) $$\|\tilde{D}\| = \|D\|_Z .$$

Oricare ar fi $G \in C_s([0,T], l^{\mathbf{z}}_{L(H,V)}))$ şi $h\in l^{\mathbf{z}}_H$, funcţia $t \in [0,T] \rightarrow \tilde{G}(t)(h) \in l^{\mathbf{z}}_{L(H,V)}$ este continuă. Deci există $M_h > 0$ astfel încât $\sup_{t\in[0,T]}\|\tilde{G}(t)(h)\|_Z < M_h$.

Aplicăm principiul mărginirii uniforme şi deducem că $\sup_{t\in[0,T]}\|\tilde{G}(t)\| < \infty$. Ţinând cont de (2) rezultă că $\sup_{s\in[0,T]}\|G(s)\|_{\mathbf{z}} < \infty$ şi de aici $|G|_T < \infty$. Este uşor de verificat că $|\cdot|_T$ este o normă pe $C_s([0,T], l^{\mathbf{z}}_{L(H,V)})$.

Acum vom arăta că $C_s([0,T], l^{\mathbf{z}}_{L(H,V)})$ este complet. Presupunem că $G_n \in C_s([0,T], l^{\mathbf{z}}_{L(H,V)})$ este un şir Cauchy în raport cu norma $|\cdot|_T$. Rezultă că pentru orice $t \in [0,T]$ există $G(t) \in l^{\mathbf{z}}_{L(H,V)}$ astfel încât $G_n(t) \rightarrow G(t)$ uniform în raport cu t dacă $n\rightarrow\infty$, adică pentru orice $\varepsilon > 0$ există $n_\varepsilon \in \mathbf{N}$ astfel încât $\|G_n(t) - G(t)\|_{\mathbf{z}} < \varepsilon$ pentru orice $n \geq n_\varepsilon, n\in\mathbf{N}$ şi $t \in [0,T]$. Mai avem de demonstrat că funcţia $t\in[0,T] \rightarrow G(t)h$ este continuă pentru orice $h \in l^{\mathbf{z}}_H$. Se observă că pentru orice $\varepsilon > 0$, $t,s \in [0,T]$ şi $h \in l^{\mathbf{z}}_H$ avem

$\|F(t)\|_T = \sup_{s\in[0,T]}\|F(s)\|$, $F \in C_s(J, L(H,V))$.

LEMMA 1. *The linear space* $C_s([0,T], l^{\mathbf{z}}_{L(H,V)}), T\in \mathbf{R}^*_+$ *is a real Banach space with the norm* $|G|_T = \sup_{s\in[0,T]}\|G(s)\|_{\mathbf{z}}$.

PROOF. For any $D \in l^{\mathbf{z}}_{L(H,V)}$, $h\in l^{\mathbf{z}}_H$, we introduce the linear and bounded operator $\tilde{D} \in L\left(l^{\mathbf{z}}_H, l^{\mathbf{z}}_{L(H,V)}\right)$ defined by $\tilde{D}$ (h) = Dh. Obviously

(2) $$\|\tilde{D}\| = \|D\|_Z .$$

For any $G \in C_s([0,T], l^{\mathbf{z}}_{L(H,V)}))$ and $h\in l^{\mathbf{z}}_H$ the mapping $t \in [0,T] \rightarrow \tilde{G}(t)(h) \in l^{\mathbf{z}}_{L(H,V)}$ is continuous. So, there is $M_h > 0$ such that $\sup_{t\in[0,T]}\|\tilde{G}(t)(h)\|_Z < M_h$.

By the Uniform Boundedness Principle we deduce that $\sup_{t\in[0,T]}\|\tilde{G}(t)\| < \infty$. From (2) it follows that $\sup_{s\in[0,T]}\|G(s)\|_{\mathbf{z}} < \infty$ and therefore $|G|_T < \infty$. It is not difficult to verify that $|\cdot|_T$ is a norm on $C_s([0,T], l^{\mathbf{z}}_{L(H,V)})$. Now let us prove that $C_s([0,T], l^{\mathbf{z}}_{L(H,V)})$ is complete. Assume that $G_n \in C_s([0,T], l^{\mathbf{z}}_{L(H,V)})$ is a $|\cdot|_T$ Cauchy sequence. It follows that for any $t \in [0,T]$ there is $G(t) \in l^{\mathbf{z}}_{L(H,V)}$ such that $G_n(t) \rightarrow G(t)$ as $n\rightarrow\infty$ uniformly with respect to *t*, i.e. for all $\varepsilon > 0$ there is $n_\varepsilon \in \mathbf{N}$ such that $\|G_n(t) - G(t)\|_{\mathbf{z}} < \varepsilon$ for all $n \geq n_\varepsilon, n\in\mathbf{N}$ and $t \in [0,T]$. We only have to prove that the mapping $t\in[0,T] \rightarrow G(t)h$ is continuous for every $h \in l^{\mathbf{z}}_H$. To this end, we observe that for any $\varepsilon > 0$, $t,s \in [0,T]$ and $h \in l^{\mathbf{z}}_H$ one has

$$\begin{aligned}&\|(G(t)-G(s))h\|_{\mathcal{Z}} \le \|G(t)-G_{n_\varepsilon}(t)\|_{\mathcal{Z}}\|h\|_{\mathcal{Z}}\\ &+\|G_{n_\varepsilon}(t)h-G_{n_\varepsilon}(s)h\|_{\mathcal{Z}}+\|G_{n_\varepsilon}(s)-G(s)\|_{\mathcal{Z}}\|h\|_{\mathcal{Z}}\\ &\le 2\varepsilon\|h\|_{\mathcal{Z}}+\|G_{n_\varepsilon}(t)h-G_{n_\varepsilon}(s)h\|_{\mathcal{Z}}.\end{aligned}$$

Cum $G_{n_\varepsilon} \in C_s([0,T], l^{\mathcal{Z}}_{L(H,V)})$, rezultă uşor că există $\delta_{\varepsilon,h,s} > 0$ astfel încât $\|G_{n_\varepsilon}(t)h - G_{n_\varepsilon}(s)h\|_{\mathcal{Z}} < \varepsilon$ pentru orice $|t-s| < \delta_{\varepsilon,h,s}$. Acum este clar că $\|(G(t)-G(s))h\|_{\mathcal{Z}} \le 2\varepsilon\|h\|_{\mathcal{Z}} + \varepsilon$ pentru orice $|t-s| < \delta_{\varepsilon,h,s}$. Rezultă concluzia.

Spaţiul funcţiilor $G : J \to L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)})$ cu proprietatea $G(X) \in C_s(J, l^{\mathcal{Z}}_{L(U,H)})$ pentru orice $X \in C_s(J, l^{z}_{E_H})$ va fi notat $C_{ss}(J, L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)}))$. Pentru subspaţiul lui $C_{ss}(J, L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)}))$ format din toate funcţiile *G* ce satisfac $\|G\|_J = \sup_{t\in J}\|G(t)\| < \infty$ folosim notaţia $C_{sb}(J, L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)}))$. Dacă $G : J \to L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)})$ şi $X : J \to l^{z}_{E_H}$ atunci elementul $G(t)(X(t))$ va fi notat uneori $G(t, X(t))$.

În cele ce urmează presupunem îndeplinită ipoteza

- (H1) $A \in C_b(\mathbf{R}_+, l^{\mathcal{Z}}_{L(H)})$, $\Pi_1(t) \in C_{sb}(\mathbf{R}_+, L(l^{z}_{E_H}))$.

În acest articol studiem problema existenţei soluţiilor de evoluţie pentru următoarea clasă de ecuaţii diferenţiale

(3)

$$\frac{dX(t)}{dt} + L(t, X(t)) + \Pi_1(t, X(t)) + Q(t) = 0,\qquad X(T) = R \in l^{z}_{E_H},$$

unde $Q \in C_s([0,T], l^{z}_{E_H})$, $R \in l^{z}_{E_H}$ şi

$$\begin{aligned}&\|(G(t)-G(s))h\|_{\mathcal{Z}} \le \|G(t)-G_{n_\varepsilon}(t)\|_{\mathcal{Z}}\|h\|_{\mathcal{Z}}\\ &+\|G_{n_\varepsilon}(t)h-G_{n_\varepsilon}(s)h\|_{\mathcal{Z}}+\|G_{n_\varepsilon}(s)-G(s)\|_{\mathcal{Z}}\|h\|_{\mathcal{Z}}\\ &\le 2\varepsilon\|h\|_{\mathcal{Z}}+\|G_{n_\varepsilon}(t)h-G_{n_\varepsilon}(s)h\|_{\mathcal{Z}}.\end{aligned}$$

Since $G_{n_\varepsilon} \in C_s([0,T], l^{\mathcal{Z}}_{L(H,V)})$, it follows easy that there is $\delta_{\varepsilon,h,s} > 0$ such that $\|G_{n_\varepsilon}(t)h - G_{n_\varepsilon}(s)h\|_{\mathcal{Z}} < \varepsilon$ for $|t-s| < \delta_{\varepsilon,h,s}$. Now it is clear that $\|(G(t)-G(s))h\|_{\mathcal{Z}} \le 2\varepsilon\|h\|_{\mathcal{Z}} + \varepsilon$ for $|t-s| < \delta_{\varepsilon,h,s}$. The conclusion follows.

The space of all mappings $G : J \to L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)})$ with the property $G(X) \in C_s(J, l^{\mathcal{Z}}_{L(U,H)})$ for all $X \in C_s(J, l^{z}_{E_H})$ will be denoted by $C_{ss}(J, L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)}))$. The subspace of $C_{ss}(J, L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)}))$ formed by all mappings *G* satisfying $\|G\|_J = \sup_{t\in J}\|G(t)\| < \infty$ we shall use the notation $C_{sb}(J, L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)}))$. If $G : J \to L(l^{z}_{E_H}, l^{\mathcal{Z}}_{L(U,H)})$ and $X : J \to l^{z}_{E_H}$ we will often use the short notation $G(t, X(t))$ for the elements $G(t)(X(t))$.

In the sequel we assume the following hypothesis

- (H1) $A \in C_b(\mathbf{R}_+, l^{\mathcal{Z}}_{L(H)})$, $\Pi_1(t) \in C_{sb}(\mathbf{R}_+, L(l^{z}_{E_H}))$.

The goal of this paper is to study the problem of existence of mild solutions for the following class of differential equations

(3)

$$\frac{dX(t)}{dt} + L(t, X(t)) + \Pi_1(t, X(t)) + Q(t) = 0,\qquad X(T) = R \in l^{z}_{E_H},$$

where $Q \in C_s([0,T], l^{z}_{E_H})$, $R \in l^{z}_{E_H}$ and $L(t)(X) = A(t)^{[*]}X + XA(t)$. The equations

$L(t)(X) = A(t)^{[*]}X + XA(t)$. Particularizând ecuaţiile de tipul (3), pe care le numim ecuaţii Lyapunov generalizate, putem obţine multe din ecuaţiile Lyapunov ce apar în studiul problemelor de stabilitate a ecuaţiilor diferenţiale liniare stochastice (EDLS).

De exemplu, în cazul în care $\sum_{j\in\mathbf{Z}, i\neq j} q_{ij} = 0$, $q_{ij} \in \mathbf{R}_+$ pentru $i \neq j$ şi $A_k \in C_b(\mathbf{R}_+, l^{\mathbf{Z}}_{\mathbf{E}_H}), B_k \in C_b(\mathbf{R}_+, l^{\mathbf{Z}}_{L(U,H)}), k = \overline{0,r}, r \in \mathbf{N}^*$ definim

$$A(t,i) = A_0(t,i) + \frac{q_{ii}}{2} I_H,$$

$$\Pi_1(t,X)(i) = \sum_{k=1}^{r} A_k^*(t,i)X(i)A_k(t,i) + \sum_{j\in\mathbf{Z}, i\neq j} q_{ij}X(j)$$

pentru orice $i \in$ **Z,** $t \in \mathbf{R}_+$ şi obţinem ecuaţia Lyapunov din [2] ce este asociată EDLS finit dimensionale cu zgomot multiplicativ şi perturbaţii Markov. Dacă $\mathbf{Z} = \{1\}$, ecuaţia Lyapunov este un caz particular al ecuaţiilor din [5], [6].

2. REZULTATUL PINCIPAL

Fie $\Delta = \{(t,s), 0 \leq s \leq t\} \subset \mathbf{R}^2$ şi pentru orice $T > 0$, fie $\Delta_T = \{(t,s), 0 \leq s \leq t \leq T\} \subset \mathbf{R}^2$.

Dacă *E* este un spaţiu Banach, atunci familia $\{U(t,s)\}_{(t,s)\in\Delta(T)} \subset L(E)$, se numeşte *operator* de *evoluţie* dacă şi numai dacă

a) $(t,s) \to U(t,s)x$ este continuu pentru orice $x \in E$ şi

b) *U(s, s) = I, U(t, s)U(s, r) = U(t, r),* pentru orice $0 \leq r \leq s \leq t \leq T$ [4].

Un *operator tare de evoluţie* (a se vedea Definiţia 4.2 din [3]) este un operator de evoluţie *U(t, s)* pentru care există o familie de operatori liniari închişi, dens definiţi *A(t), t ≥ 0* , cu domeniul *D(A(t))* astfel încât

s_1) $U(t,s) : D(A(s)) \to D(A(t))$ pentru $t > s$

s_2) $\frac{\partial U(t,s)x}{\partial t} = A(t)U(t,s)x$ pentru orice $x \in D(A(s))$ şi toţi $t > s \geq 0$.

of type (3), which we called generalized Lyapunov equations, include as special cases many of the Lyapunov equations involved in stability problems for linear stochastic differential equations (LSDEs).

For example, in the case when $\sum_{j\in\mathbf{Z}, i\neq j} q_{ij} = 0$, $q_{ij} \in \mathbf{R}_+$ for $i \neq j$ and $A_k \in C_b(\mathbf{R}_+, l^{\mathbf{Z}}_{\mathbf{E}_H}), B_k \in C_b(\mathbf{R}_+, l^{\mathbf{Z}}_{L(U,H)}), k = \overline{0,r}, r \in \mathbf{N}^*$ we define

$$A(t,i) = A_0(t,i) + \frac{q_{ii}}{2} I_H,$$

$$\Pi_1(t,X)(i) = \sum_{k=1}^{r} A_k^*(t,i)X(i)A_k(t,i) + \sum_{j\in\mathbf{Z}, i\neq j} q_{ij}X(j)$$

For all $i \in Z$, $t \in R_+$ and we recover the Lyapunov equation from [2] associated with finite dimensional LSDEs with multiplicative noise and Markovian switches. If $\mathbf{Z} = \{1\}$, the Lyapunov equation is a special case of the ones in [5], [6].

2. MAIN RESULTS

Let $\Delta = \{(t,s), 0 \leq s \leq t\} \subset \mathbf{R}^2$ and for any $T > 0$, let $\Delta_T = \{(t,s), 0 \leq s \leq t \leq T\} \subset \mathbf{R}^2$.

If *E* is a Banach space, then a family $\{U(t,s)\}_{(t,s)\in\Delta(T)} \subset L(E)$, is called an *evolution operator* (system) iff

a) $(t,s) \to U(t,s)x$ is continuous for any $x \in E$ and

b) *U(s, s) = I, U(t, s)U(s, r) = U(t, r),* for all $0 \leq r \leq s \leq t \leq T$ [4].

A *strong evolution* operator (see Definition 4.2 in [3]) is an evolution operator *U(t, s)* for which there is a closed linear and densely defined operator *A(t), t ≥ 0* , with domain *D(A(t))* such that

s_1) $U(t,s) : D(A(s)) \to D(A(t))$ for $t > s$

s_2) $\frac{\partial U(t,s)x}{\partial t} = A(t)U(t,s)x$ for every $x \in D(A(s))$ and all $t > s \geq 0$.

In this case we say that the family *A(t), t ≥ 0* generates the strong evolution operator *U(t, s).*

În acest caz spunem că familia *A(t), t ≥ 0* generează operatorul tare de evoluţie *U(t, s).* Operatorul de evoluţie *U(t, s)* se numeşte *operator aproape tare de evoluţie* dacă satisface condiţiile (s_1) şi (s_2) numai pentru aproape toţi $t \geq s$.

Fie $B^\infty(\mathbf{R}_+, L(H))$ spaţiul tuturor funcţiilor tare măsurabile $f : \mathbf{R}_+ \to L(H)$ [7] cu proprietatea că $\operatorname{ess\,sup}_{0\le t\le T}\|f(t)\| < \infty$ pentru orice *T > 0* şi fie *A* familia definită de ipoteza (H1). Atunci, pentru orice $i \in \mathbf{Z}$, $A(.,i) \in C_s(\mathbf{R}_+, L(H))$ şi $A^*(.,i) \in B^\infty(\mathbf{R}_+, L(H))$. Este cunoscut faptul că $A(.,i)$ (respectiv $A^*(.,i)$) generează un operator de evoluţie tare (respectiv aproape tare) $\{U(t,s;i),(t,s) \in \Delta\}$ (respectiv $\{V(t,s;i),(t,s) \in \Delta\}$) pe *H* (a se vedea Lema 3 din [5]).

Folosind metoda standard se poate arăta că:

u_1) $\|U(t,s;i)\| \le e^{\lambda(t-s)}, \|U(t,s;i) - I\| \le \lambda e^{\lambda T}(t-s)$ pentru orice $i \in \mathbf{Z}$, unde $\lambda = \sup_{t\in\mathbf{R}_+}\|A(t)\|_{\mathbf{Z}}$;

u_2) $(t,s) \to U(t,s;i)$ este $\|.\|-$ continuu oe Δ, uniform în raport cu $i \in \mathbf{Z}$;

u_3) $U(t,s;i)$ este tare diferenţiabil în a doua variabilă şi pentru orice $i \in \mathbf{Z}, x \in H$ avem

$$\frac{\partial U(t,s;i)x}{\partial s} = -U(t,s;i)A(s,i)x\,;$$

u_4) $U^{-1}(t,s;i)$ există pentru orice $(t,s) \in \Delta$ şi $i \in \mathbf{Z}$ şi $U^{-1}(t,s;i)$ = U(s,t;i) [1].

Acum, pentru orice $t \in \mathbf{R}_+$ şi $(t,s) \in \Delta$, asociem lui $A(t)$ următorii operatori liniari:

$$L(t)(X) = A(t)^{[*]}X + XA(t),$$
$$\Phi(t,s)(X) = U(t,s)^{[*]}XU(t,s), X \in l^{\mathbf{z}}_{\mathbf{E}_H}$$

Este uşor de văzut că $L(t), \Phi(t,s) \in L(l^{\mathbf{z}}_{\mathbf{E}_H})$. În plus, din proprietăţile lui $U(t,s;i), i \in \mathbf{Z}$, deducem că:

f_1) $\Phi(t,t) = I_{l^{\mathbf{z}}_{\mathbf{E}_H}}$;

The evolution operator *U(t, s)* is called an *almost strong evolution operator* if it satisfies conditions (s_1) and (s_2) only for almost all $t \geq s$.

Let $B^\infty(\mathbf{R}_+, L(H))$ be the space of all strongly measurable mappings $f : \mathbf{R}_+ \to L(H)$ [7] such that $\operatorname{ess\,sup}_{0\le t\le T}\|f(t)\| < \infty$ for all *T > 0* and let *A* be the family defined by the hypothesis (H1). Then, for every $i \in \mathbf{Z}$, $A(.,i) \in C_s(\mathbf{R}_+, L(H))$ and $A^*(.,i) \in B^\infty(\mathbf{R}_+, L(H))$. It is known that $A(.,i)$ (respectively $A^*(.,i)$) generates a strong (respectively almost strong) evolution operator $\{U(t,s;i),(t,s) \in \Delta\}$ (respectively $\{V(t,s;i),(t,s) \in \Delta\}$) on *H* (see Lemma 3 in [5]).

By a standard way it can be proved that:

u_1) $\|U(t,s;i)\| \le e^{\lambda(t-s)}, \|U(t,s;i) - I\| \le \lambda e^{\lambda T}(t-s)$ for all $i \in \mathbf{Z}$, where $\lambda = \sup_{t\in\mathbf{R}_+}\|A(t)\|_{\mathbf{Z}}$;

u_2) $(t,s) \to U(t,s;i)$ is $\|.\|-$ continuous on Δ, uniformly with respect to $i \in \mathbf{Z}$;

u_3) $U(t,s;i)$ is strongly differentiable in the second variable and for all $i \in \mathbf{Z}, x \in H$ we have $\frac{\partial U(t,s;i)x}{\partial s} = -U(t,s;i)A(s,i)x$;

u_4) $U^{-1}(t,s;i)$ exists for all $(t,s) \in \Delta$ and $i \in \mathbf{Z}$ and $U^{-1}(t,s;i)$ = U(s,t;i) [1].

Now, for all $t \in \mathbf{R}_+$ and $(t,s) \in \Delta$, we associate with $A(t)$ the following linear operators:

$$L(t)(X) = A(t)^{[*]}X + XA(t),$$
$$\Phi(t,s)(X) = U(t,s)^{[*]}XU(t,s), X \in l^{\mathbf{z}}_{\mathbf{E}_H}$$

It is easy to see that $L(t), \Phi(t,s) \in L(l^{\mathbf{z}}_{\mathbf{E}_H})$. Moreover, from the properties of $U(t,s;i), i \in \mathbf{Z}$, we know that:

f_1) $\Phi(t,t) = I_{l^{\mathbf{z}}_{\mathbf{E}_H}}$;

f_2) $\Phi(r,s)\Phi(t,r) = \Phi(t,s)$ for all

f_2) $\Phi(r,s)\Phi(t,r) = \Phi(t,s)$ pentru orice $0 \le s \le r \le t$;

f_3) $(t,s) \to \Phi(t,s)$ is $\|.\|$ -continuu pe $\Delta_T, T > 0$;

f_4) $\Phi(t,s)(\mathbf{K}_H^{\mathbf{Z}}) \subset \mathbf{K}_H^{\mathbf{Z}}$, adică $\Phi(t,s)$ este un operator pozitiv.

Deci $\Phi(t,s)$ este un operator de evoluţie, pozitiv, pe care îl numim *evoluţie pozitivă.* Deşi nu putem arăta că $\Phi(t,s)$ este tare diferenţiabil în t sau s pe $l_{\mathbf{E}_H}^{\mathbf{Z}}$, din (s_2) şi (u_3), deducem următoarea proprietate de diferenţiabilitate pe componente a lui $\Phi(t,s)$:

f_5) pentru orice $x, y \in H$, $X \in l_{\mathbf{E}_H}^{\mathbf{Z}}$ avem

$$\frac{\partial \Phi(t,s)(X)(i)x}{\partial s} = -L(s, \Phi(t,s)(X))(i)x$$

aproape peste tot pe [0, t] şi

$$\frac{\partial \langle \Phi(t,s)(X)(i)x, y\rangle}{\partial t} = \langle \Phi(t,s)(L(t,X))(i)x, y\rangle$$

pentru orice $s \le t$.

Ecuaţiei Lyapunov (3) îi asociem ecuaţia integrală

(4)

$$X(s)x = \Phi(T,s)(R)h + + \int_s^T \Phi(r,s)(\Pi_1(r, X(r)) + Q(r))hdr, h \in l_H^{\mathbf{Z}},$$

$s \le T$.

Prin soluţie de evoluţie pe $[0,T]$ a ecuaţiei (3) înţelegem un element $X(s) = X(T,.,R)$ al spaţiului $C_s([0,T], l_{\mathbf{E}_H}^{\mathbf{Z}})$ care satisface (4).

TEOREMA 1 *a) În ipotezele de mai sus ecuaţia (4) are o soluţie unică ce satisface ecuaţia integrală de mai jos*

(5)

$$X(s)h = \Phi(t,s)(X(t))h + + \int_s^t \Phi(r,s)(\Pi_1(r, X(r)) + Q(r))hdr,$$

pentru orice $s \le t \le T$ *şi* $h \in l_H^{\mathbf{Z}}$.

b) Funcţiile $s \to X(T,s;R)$ *şi*

$0 \le s \le r \le t$;

f_3) $(t,s) \to \Phi(t,s)$ is $\|.\|$ -continuous on $\Delta_T, T > 0$;

f_4) $\Phi(t,s)(\mathbf{K}_H^{\mathbf{Z}}) \subset \mathbf{K}_H^{\mathbf{Z}}$, i.e $\Phi(t,s)$ is a positive operator.

Therefore $\Phi(t,s)$ is a positive evolution operator, which we call a *positive evolution.* Although we cannot prove that $\Phi(t,s)$ is strongly differentiable in t or s on $l_{\mathbf{E}_H}^{\mathbf{Z}}$, by (s_2) and (u_3), we have the following componetwise differentiability of $\Phi(t,s)$:

f_5) for all $x, y \in H$, $X \in l_{\mathbf{E}_H}^{\mathbf{Z}}$ we have

$$\frac{\partial \Phi(t,s)(X)(i)x}{\partial s} = -L(s, \Phi(t,s)(X))(i)x$$

and

a.e.on $[0,t]$

$$\frac{\partial \langle \Phi(t,s)(X)(i)x, y\rangle}{\partial t} = \langle \Phi(t,s)(L(t,X))(i)x, y\rangle,$$

for all $s \le t$.

With the formal Lyapunov equation (3) we associate the integral equation

(4)

$$X(s)x = \Phi(T,s)(R)h + + \int_s^T \Phi(r,s)(\Pi_1(r, X(r)) + Q(r))hdr, h \in l_H^{\mathbf{Z}},$$

$s \le T$.

By a mild solution $X(s) = X(T,.,R)$ on $[0,T]$ of (3) we mean an element of $C_s([0,T], l_{\mathbf{E}_H}^{\mathbf{Z}})$ which satisfies (4).

THEOREM 1 *a) Under the above hypotheses the equation (4) has a unique mild solution which also satisfies*

(5)

$$X(s)h = \Phi(t,s)(X(t))h + + \int_s^t \Phi(r,s)(\Pi_1(r, X(r)) + Q(r))hdr,$$

for all $s \le t \le T$ *and* $h \in l_H^{\mathbf{Z}}$.

b) The mappings $s \to X(T,s;R)$ *and*

$T \to X(T,s;R), 0 \le s \le T$ *sunt* $\|.\|_Z$ *continue, uniform în raport cu T şirespectiv s.*

DEMONSTRAŢIE. Din Lema 1 deducem că, pentru orice $\tau \in [0,T]$, $C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$ este spaţiu Banach împreună cu norma $|G|_{T,\tau} = \sup_{s\in[T-\tau,T]} \|G(s)\|_{\mathbf{z}}$.

Pentru orice $X \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$, $h \in l^{\mathbf{z}}_H$, definim

(6)
$$F(X)(s)h = \Phi(T,s)(R)h + \int_s^T \Phi(r,s)(\Pi_1(r,X(r)) + Q(r))hdr.$$

Vom arăta că *F* este bine definită. Din (H1) şi (1), deducem că funcţia

$$r \to U^{[*]}(r,s)[\Pi_1(r,X(r)) + Q(r)]U(r,s)h \in l^{\mathbf{z}}_H$$

este continuă, deci Bochner măsurabilă, şi mărginită pe $[T-\tau,T]$ pentru orice $X \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$. Rezultă că ea este Bochner integrabilă (a se vedea [7]) şi integrala din (6) este bine definită. Pe de altă parte $s \to U^{[*]}(r,s)[\Pi_1(r,X(r)) + Q(r)]U(r,s)$ este $\|.\|_{\mathbf{z}}$- continuă, uniform în raport cu *r* , şi de aici se deduce uşor că $F(X) \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$.

Acum vom arăta că F este o contracţie pe $C_s([0,T], l^{\mathbf{z}}_{\mathbf{E}_H})$. Pentru orice $X, Y \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$ şi $s \in [T-\tau,T]$ avem

$T \to X(T,s;R), 0 \le s \le T$ *are* $\|.\|_Z$ *continuous, uniformly with respect to T and s respectively.*

PROOF. From Lemma 1 we deduce that, for any $\tau \in [0,T]$, $C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$ is a Banach space when endowed with the norm $|G|_{T,\tau} = \sup_{s\in[T-\tau,T]} \|G(s)\|_{\mathbf{z}}$.

For all $X \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$, $h \in l^{\mathbf{z}}_H$, we define the mapping

(6)
$$F(X)(s)h = \Phi(T,s)(R)h + \int_s^T \Phi(r,s)(\Pi_1(r,X(r)) + Q(r))hdr.$$

Let us prove that *F* is well defined. From (H1) and (1), the function

$$r \to U^{[*]}(r,s)[\Pi_1(r,X(r)) + Q(r)]U(r,s)h \in l^{\mathbf{z}}_H$$

is continuous (hence Bochner measurable) and bounded on $[T-\tau,T]$ for any $X \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$. Therefore it is Bochner integrable (see [7]) and the integral from (6) exists. On the other hand $s \to U^{[*]}(r,s)[\Pi_1(r,X(r)) + Q(r)]U(r,s)$ is $\|\cdot\|_{\mathbf{z}}$ continuous, uniformly with respect to *r* , and we deduce easily that $F(X) \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$.

Now we show that F is a contraction on $C_s([0,T], l^{\mathbf{z}}_{\mathbf{E}_H})$. For any $X, Y \in C_s([T-\tau,T], l^{\mathbf{z}}_{\mathbf{E}_H})$ and $s \in [T-\tau,T]$ we have

$$\|F(X)(s)h - F(Y)(s)h\|_{\mathbf{z}} \le \int_s^T \|\Phi(r,s)(\Pi_1(r,X(r)-Y(r)))h\|_{\mathbf{z}} dr$$
$$\le \int_s^T \|\Phi(r,s)\| \|\Pi_1(r,X(r)-Y(r))\|_{\mathbf{z}} \|h\|_{\mathbf{z}} dr \le \|\Pi_1\|_{\mathbf{R}_+} \int_s^T e^{2\lambda(r-s)} \|X(r)-Y(r)\|_{\mathbf{z}} \|h\|_{\mathbf{z}} dr$$
$$\le e^{2\lambda T} \|\Pi_1\|_{\mathbf{R}_+} |X-Y|_{T,\tau} \|h\|_{\mathbf{z}} (T-s)$$

Deci există $M_T > 0$ depinzând de *T* astfel încât

Therefore there is M_T depending on *T* such that

$$\|F(X)(s)-F(Y)(s)\| \le M_T|X-Y|_{T,\tau}(T-s)$$

pentru orice $s \in [T-\tau, T]$. Rezultă că $|F(X)-F(Y)|_{T,\tau} \le M_T|X-Y|_{T,\tau}\tau$. Pentru τ cu proprietatea $\tau M_T < 1$ funcţia *F* este o contracţie.

Aplicând teorema de punct fix a lui Banach, deducem că ecuaţia (4) are o unică soluţie de evoluţie X_0 pe $[T-\tau, T]$. Un calcul direct arată că X_0 satisface (5) pe $[T-\tau, T]$. Înlocuind *T* şi *R* cu $T-\tau$ şi respectiv $X(T-\tau)$, se poate arăta că (4) are o unică soluţie de evoluţie X_1 pe $[T-2\tau, T-\tau]$. Ţinând cont de (5) şi de rezultatele de mai sus, deduce că funcţia

$$X_{01}(s) = \begin{cases} X_0(s), s \in [T-\tau, T] \\ X_1(s), s \in [T-2\tau, T-\tau] \end{cases},$$

este unica soluţie de evoluţie a ecuaţiei (4) în $C_s([T-2\tau, T], l^{\mathbf{z}}_{\mathbf{E}_H})$. (Aceasta deoarece soluţia de evoluţie (unică) a lui (4), pe $[T-3\tau/2, T-\tau/2]$, ce satisface condiţia $X(T-\tau/2) = X_0(T-\tau/2)$ coincide cu X_0 pe $[T-\tau, T-\tau/2]$ şi cu X_1 pe $[T-3\tau/2, T-\tau]$.) Repetând raţionamentul se obţine o soluţie de evoluţie a ecuaţiei (4) pe $[0, T]$ ce satisface (5). Din unicitatea soluţiei de evoluţie a ecuaţiei (4) pe $[0, T]$ rezultă concluzia. Pentru a demonstra prima parte a afirmaţiei b), observăm că dacă $s > s_0$ avem

$$\|F(X)(s)-F(Y)(s)\| \le M_T|X-Y|_{T,\tau}(T-s)$$

for all $s \in [T-\tau, T]$. It follows that $|F(X)-F(Y)|_{T,\tau} \le M_T|X-Y|_{T,\tau}\tau$. For τ such that $\tau M_T < 1$ the mapping *F* is a contraction.

Applying the Banach fixed point theorem, we deduce that equation (4) has a unique mild solution X_0 on $[T-\tau, T]$. A direct computation shows that X_0 verifies (5) on $[T-\tau, T]$. Replacing *T* and *R* with $T-\tau$ and $X(T-\tau)$, respectively, it can be proved that (4) has a unique mild solution X_1 on $[T-2\tau, T-\tau]$. In view of (5) and the above results, the mapping

$$X_{01}(s) = \begin{cases} X_0(s), s \in [T-\tau, T] \\ X_1(s), s \in [T-2\tau, T-\tau] \end{cases},$$

is the unique mild solution of (4) in $C_s([T-2\tau, T], l^{\mathbf{z}}_{\mathbf{E}_H})$. (This is because the unique solution of (4) on $[T-3\tau/2, T-\tau/2]$ satisfying $X(T-\tau/2) = X_0(T-\tau/2)$ coincides with X_0 on $[T-\tau, T-\tau/2]$ and with X_1 on $[T-3\tau/2, T-\tau]$.) Repeating the procedure we get a unique solution of (4) on $[0, T]$ which also satisfies (5). For the first part of statement b), we see that if $s > s_0$ we have

$$\begin{aligned} \|(X(s)-X(s_0))\|_{\mathbf{z}} \le {} & \|\Phi(T,s)-\Phi(T,s_0)\|\|R\|_{\mathbf{z}} + \\ & + \int_s^T \|\Phi(r,s)-\Phi(r,s_0)\|\left(\|\Pi_1\|_{\mathbf{R}_+}|X|_T + |Q|_T\right)dr \\ & + \int_{s_0}^{s} \|\Phi(r,s_0)\|\left(\|\Pi_1\|_{\mathbf{R}_+}|X|_T + |Q|_T\right)dr. \end{aligned}$$

Aplicând $f_3)$ obţinem concluzia. Cazul $s < s_0$ precum şi ultima parte a afirmaţiei b) se demonstrează asemănător.

Applying $f_3)$ we get the conclusion. The case $s < s_0$ and the last part of b) follows similarly.

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