THE VARIATIONAL METHOD OF SCHIFFER-GOLUZIN IN AN EXTREMAL PROBLEM OF CLASS S

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ABSTRACT: Let S be the class of analytic functions of the form $f(z)=z+a_2z^2+\cdots, f(0)=0, f'(0)=1$ defined on the unit disk $|z|<1$. Petru T. Mocanu [2] raised the question of determination $\max_{f\in S} |f(z)|$ which satisfies the conditions $|f'(z)|=1$, $|z|=r$, $f\in S$, $0\le r<1$, r given. For solving the problem we shall use the variational method of Schiffer-Goluzin [1].

KEY WORDS: class of function, analitic

1. INTRODUCTION

Let S be the class of functions

$f(z)=z+ a_2z^2+\ldots,$

$f(0)=0$, $f'(0)=1$ which are regular and univalent in the unit disk $|z|<1$. In [1] Petru T. Mocanu raised the issue of determination

$$\max_{f\in S} |f(z)| \qquad (1)$$

$$|f'(z)|=1, \ |z|=r, \ f\in S, \ 0\le r<1, r \text{ given.} \qquad (2)$$

Since S is a compact class (1) is attained. The aim of this paper is to solve the problem by using the variational method of Schiffer-Goluzin [2].

2. CONTENTS

Let $f\in S$ the extremal function for which (1) is attained in conditions (2) . In order to solve this problem let us consider a variation $f^*(z)$ of the function $f(z)$ given by the Schiffer-Goluzin's formula :

$$f^*(z) = f(z)+\lambda V(z;\zeta;\psi)+O(\lambda^2), \ |\zeta|<1, \ \lambda>0, \ \psi \text{ real} \qquad (3)$$

where

$$V(z,\zeta;\psi) = e^{i\psi}\frac{f^2(z)}{f(z)-f(\zeta)} - e^{i\psi}f(z)\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 - e^{i\psi}\frac{zf'(z)}{z-\zeta}\zeta\left[\frac{f(\zeta)}{\zeta f'(z)}\right]^2 + e^{-i\psi}\frac{z^2f'(z)}{1-\overline{\zeta}z}\overline{\zeta}\overline{\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2} \qquad (4)$$

It is known that if λ is small enough, the function $f^*(z)$ belongs to S class. Let us consider a variation z^* of z :

$$z^* = z+\lambda h+O(\lambda^2), \quad h = \frac{\partial z^*}{\partial\lambda}\Big|_{\lambda=0}$$

which satisfies the conditions :

$$|z^*| = r \text{ and } |f^{*\prime}(z^*)| = 1 \qquad (5)$$

We notice that

$$|z^*|^2 = |z|^2+2\lambda\,\mathrm{Re}(\overline{z}h)+O(\lambda^2) = r^2.$$

Since $|z|=r$ from the relation above we obtain $\mathrm{Re}(\overline{z}h) = 0$. Replacing z with z^* in relation (3) we have

$$f^{*\prime}(z^*) = f'(z^*)+\lambda V'(z^*;\zeta;\psi)+O(\lambda^2).$$

The condition $|f'(z)|=1$ becomes:

$$2\operatorname{Re}\left\{\overline{f'(z)}\left[hf''(z)+V'(z;\zeta;\psi)\right]\right\} = 0 \qquad (6)$$

Since f(z) is extremal |f *(z*)|≤| f(z) | which is equivalent with:

$$\left|f(z)+\lambda hf'(z)+\cdots+\lambda V(z;\zeta;\psi)+\cdots\right| \le \left|f(z)\right|. \qquad (7)$$

By squaring in (7) and using the equality $u\overline{u} = |u|^2$:

$$\left\{f(z)+\lambda\left[hf'(z)+V(z;\zeta;\psi)\right]+\cdots\right\}\times \left\{\overline{f(z)}+\lambda\left[\overline{h}\,\overline{f'(z)}+\overline{V(z;\zeta;\psi)}\right]+\cdots\right\} \le f(z)\overline{f(z)}. \qquad (8)$$

From relation (8) we obtain the condition :

$$2\operatorname{Re}\left\{\overline{f(z)}\left[hf'(z)+V(z;\zeta;\psi)\right]\right\} \le 0. \qquad (9)$$

From condition $\operatorname{Re}(\overline{z}h)=0.$ we note that $\overline{h} = -\frac{\overline{z}}{z}h$; having this the relation (6) can also be written as it follows :

$$\overline{f'(z)}\left[hf''(z)+V'(z;\zeta;\psi)\right]+f'(z)\left[-\frac{\overline{z}}{z}h\overline{f''(z)}+\overline{V'(z;\zeta;\psi)}\right]=0$$

where

$$h = \frac{z\overline{f'(z)}V'(z;\zeta;\psi)+zf'(z)\overline{V'(z;\zeta;\psi)}}{\overline{z}f'(z)\overline{f''(z)}-z\overline{f'(z)}f''(z)} \qquad (10)$$

Next we use the following notations

$$f=f(z), w=f(\zeta), l=f'(z), m=f''(z), V=V(z;\zeta;\psi), V'=V'_z(z;\zeta;\psi).$$

By using the notations above and h from relation (10) relation (9) can be written as it follows :

$$\operatorname{Re}\left[\overline{f}\left(plV'+V\right)\right]\le 0 \qquad (11)$$

where

$$p = \frac{zl-\overline{z}\,\overline{l}}{\overline{z}l\overline{m}-z\overline{l}m}$$, p real .

1°.We assume that $\operatorname{Im}(z\overline{l}m)\ne 0 \quad (\overline{z}l\overline{m}-z\overline{l}m\ne 0)$. From relation (4) we obtain:

$$V = e^{i\psi}\frac{f^2}{f-w}-e^{i\psi}f\left(\frac{w}{\zeta w'}\right)^2-e^{i\psi}\frac{zl}{z-\zeta}\zeta\left(\frac{w}{\zeta w'}\right)^2+e^{-i\psi}\frac{z^2l}{1-\overline{\zeta}z}\overline{\zeta}\overline{\left(\frac{w}{\zeta w'}\right)^2}$$

and

$$V' = e^{i\psi}\frac{fl(f-2w)}{(f-w)^2}-e^{i\psi}l\left(\frac{w}{\zeta w'}\right)^2-e^{i\psi}\frac{z(z-\zeta)m-\zeta l}{(z-\zeta)^2}\zeta\left(\frac{w}{\zeta w'}\right)^2+ e^{-i\psi}\frac{z^2(1-\overline{\zeta}z)m+zl(2-\overline{\zeta}z)}{(1-\overline{\zeta}z)^2}\overline{\zeta}\overline{\left(\frac{w}{\zeta w'}\right)^2}.$$

By replacing the expresions of V and V ′ in (11) we obtain the relation

$$\operatorname{Re}\left[e^{i\psi}(E-GF)\right]\le 0 \qquad (12)$$

where

$$E = \frac{f\overline{f}[(-2pl^2-f)w+pl^2f+f^2]}{(f-w)^2}, \quad F=\left(\frac{w}{\zeta w'}\right)^2$$

and

$$G = pl^2\overline{f}+\frac{pl\overline{f}[z(z-\zeta)m-\zeta l]}{(z-\zeta)^2}\zeta-\frac{p\overline{l}f[\overline{z}^2(1-\zeta\overline{z})\overline{m}+\overline{z}\,\overline{l}(2-\zeta\overline{z})]}{(1-\zeta\overline{z})^2}\zeta+ f\overline{f}+\frac{zl\overline{f}}{z-\zeta}\zeta-\frac{\overline{z}^2\overline{l}f}{1-\zeta\overline{z}}\zeta.$$

Since ψ is arbitrary, from relation (12) it results that the extremal function w=f(ζ) must satisfy the following differential equation:

$$\left(\frac{\zeta w'}{w}\right)^2\frac{f\overline{f}\left[(-2pl^2-f)w+pl^2f+f^2\right]}{(f-w)^2} = \frac{\sum_{s=0}^{4}t_s\zeta^s}{(z-\zeta)^2(1-\zeta\overline{z})^2} \qquad (13)$$

where the coefficients t_s , $s\in\{0,1,2,3,4\}$ have the following expressions

$$t_0 = z^2\overline{f}(pl^2+f);$$
$$t_1 = -2f\overline{f}z(1+2r^2)-l\overline{f}z^2-\overline{l}fr^4-2pl^2\overline{f}z(1+r^2)- pl\overline{f}mz^2-p\overline{l}f(\overline{z}^2\overline{m}+2\overline{l}zr^2);$$
$$t_2 = f\overline{f}(r^4+4r^2+1)-2zl\overline{f}(2r^2+1)+\overline{zl}fr^2(r^2+2)+ +pl^2f(r^4+4r^2+1)-pl\overline{f}[2mz(r^2+1)+l]-2r^2(\overline{mz}+2\overline{l});$$
$$t_3 = -\overline{z}(2r^2+1)\left(2f\overline{f}+\overline{lz}f+2pl^2\overline{f}\right)+l\overline{f}r^2(r^2+2)+ pl\overline{f}(mr^4+2mr^2+2l\overline{z});$$
$$t_4 = f\overline{f}\overline{z}^2-\overline{lz}fr^2+\overline{l}f\overline{z}^3+pl^2\overline{f}\overline{z}^2-pl\overline{f}\overline{z}^2(mz+l)+p\overline{l}f\overline{z}^2\overline{mz}+l.$$

The extremal function turns the unit disk into a domain without exterior points. In order to justify this it is sufficient to assume that the domain transformed by an extremal function $w=f(\zeta)$ has an exterior point w_0 and to consider the function of variation :

$$f^*(z) = f(z) + \lambda e^{i\psi}\frac{f^2(z)}{f(z)-w_0}$$, $\lambda>0$, ψ real

which belongs to class S .

3. It si known that the extremal function $w=f(\zeta)$ transforms the unit disk $|\zeta|<1$,onto the entire plane, slit along a finite number of analytic arcs.

Let $q=e^{10}$ be the point on the circle $|\zeta|=1$ which corresponds to the extremity of such slits in which $w'(q)=0$ and $\zeta=q$ double root for the polynomial $\sum_{s=0}^{4} t_s\zeta^s$.

Since $\zeta=q$ is a double root for this polynomial, we can write:

$$\sum_{s=0}^{4} t_s\zeta^s = (1-\overline{q}\zeta)^2(a_0+a_1\zeta+a_2\zeta^2).$$

From the expressions of the coefficients t_s , $s\in\{0,1,2,3,4\}$ it results that we can consider $a_0=t_0, a_1=-2kq$, k real and $a_2=q^2t_4$.

The differential equation (13) can also be written as it follows:

$$\left(\frac{\zeta w'}{w}\right)^2\frac{f\overline{f}[(-2pl^2-f)w+pl^2f+f^2]}{(f-w)^2} = \frac{(1-\overline{q}\zeta)^2(t_0-2kq\zeta+q^2t_4\zeta^2)}{(z-\zeta)^2(1-\overline{z}\zeta)^2} \qquad (14)$$

4. From the differential equation (14) we obtain :

$$\frac{\sqrt{f\overline{f}[(-2pl^2-f)w+pl^2f+f^2]}}{w(f-w)}dw = \frac{(1-\overline{q}\zeta)\sqrt{t_0-2kq\zeta+q^2t_4\zeta^2}}{\zeta(z-\zeta)(1-\overline{z}\zeta)}d\zeta$$

where

$$\int_0^w \frac{\sqrt{f\overline{f}[(-2pl^2-f)w+pl^2f+f^2]}}{w(f-w)}dw = \int_0^\zeta \frac{(1-\overline{q}\zeta)\sqrt{t_0-2kq\zeta+q^2t_4\zeta^2}}{\zeta(z-\zeta)(1-\overline{z}\zeta)}d\zeta \qquad (15)$$

For the computation of the integral on the left of the relation (15) we note

$$I_1 = \int_0^w \frac{\sqrt{f\overline{f}[(-2pl^2-f)w+pl^2f+f^2]}}{w(f-w)}dw.$$

We can write

$$I_1 = \sqrt{f\overline{f}(-2pl^2-f)}\int\frac{\sqrt{w+\alpha^2}}{w(f-w)}dw$$ where

$$\alpha^2 = \frac{pl^2f+f^2}{-2pl^2-f}.$$

We notice that $w+\alpha^2=u^2$; I_1 becomes:

$$I_1 = \sqrt{f\overline{f}(-2pl^2-f)}\int\frac{-2u^2du}{(u^2-\alpha^2)(u^2-\beta^2)} \quad (dw=2udu, \beta^2=\alpha^2+f)$$

We notice that

$$\frac{-2u^2}{(u^2-\alpha^2)(u^2-\beta^2)} = \frac{2\alpha^2}{\beta^2-\alpha^2}\frac{1}{u^2-\alpha^2} - \frac{2\beta^2}{\beta^2-\alpha^2}\frac{1}{u^2-\beta^2} \quad ,\beta^2-\alpha^2=f$$

and the integral above becomes:

$$I_1 = \frac{\sqrt{f\overline{f}(-2pl^2-f)}}{f}\left(\alpha\ln\frac{u-\alpha}{u+\alpha} - \beta\ln\frac{u-\beta}{u+\beta}\right)$$

or

$$I_1 = \frac{\sqrt{f\overline{f}(-2pl^2-f)}}{f}\ln\left[\left(\frac{u-\alpha}{u+\alpha}\right)^\alpha\left(\frac{u+\beta}{u-\beta}\right)^\beta\right] \qquad (16)$$

In order to compute the integral on the right of the relation (15) we note

$$I_2 = \int\frac{(1-\overline{q}\zeta)\sqrt{t_0-2kq\zeta+q^2t_4\zeta^2}}{\zeta(z-\zeta)(1-\overline{z}\zeta)}d\zeta$$

We notice that $t_0-2kq\zeta+q^2t_4\zeta^2 = q^2t_4(\zeta-\zeta_1)(\zeta-\zeta_2)$ where $\zeta_{1,2} = \frac{k\pm\sqrt{k^2-t_0t_4}}{t_4}\overline{q}$.

If we note $k-\sqrt{k^2-t_0t_4}=\rho$, we notice that $\zeta_1=\frac{\rho}{t_4}\overline{q}$ and $\zeta_2=\frac{t_0}{\rho}\overline{q}$.Using the notations above, we can write: $\sqrt{t_0-2kq\zeta+q^2t_4\zeta^2} = \sqrt{q^2t_4}\sqrt{\left(\zeta-\frac{\rho}{t_4}\overline{q}\right)\left(\zeta-\frac{t_0}{\rho}\overline{q}\right)}$.

In order to compute integral I_2 we make the following substitution:

$$\sqrt{\left(\zeta - \frac{\rho}{t_4}\overline{q}\right)\left(\zeta - \frac{t_0}{\rho}\overline{q}\right)} = \nu\left(\zeta - \frac{\rho}{t_4}\overline{q}\right). \quad (17)$$

From (17) we obtain:

$\zeta = \sigma\frac{\nu^2 - a^2}{\nu^2 - 1}$ where $\sigma = \frac{\rho}{t_4}\overline{q}$ and $a^2 = \frac{t_0 t_4}{\rho^2}$ (18)

Next,

$$\begin{cases} d\zeta = \dfrac{2\sigma(a^2-1)}{(\nu^2-1)^2}d\nu, \\ z-\zeta = (z-\sigma)\dfrac{\nu^2-b^2}{\nu^2-1} \text{ with } b^2 = \dfrac{\sigma a^2 - z}{\sigma - z}, \end{cases} \quad (19)$$

and

$$\begin{cases} 1-\overline{z}\zeta = (1-\overline{z}\sigma)\dfrac{\nu^2-c^2}{\nu^2-1} \text{ with } c^2 = \dfrac{1-\overline{z}\sigma a^2}{1-\overline{z}\sigma} \\ 1-\overline{q}\zeta = (1-\overline{q}\sigma)\dfrac{\nu^2-d^2}{\nu^2-1} \text{ with } d^2 = \dfrac{1-\overline{q}\sigma a^2}{1-\overline{q}\sigma}, \\ \sqrt{\left(\zeta - \dfrac{\rho}{t_4}\overline{q}\right)\left(\zeta - \dfrac{t_0}{\rho}\overline{q}\right)} = \dfrac{\sigma(1-a^2)\nu}{\nu^2-1}. \end{cases} \quad (20)$$

By using the relations above we obtain:

$$I_2 = \frac{2\sigma q(1-\overline{q}\sigma)(1-a^2)^2\sqrt{t_4}}{(\sigma - z)(1-\overline{z}\sigma)}\int P(\nu)d\nu, \quad (21)$$

where

$$P(\nu) = \frac{\nu^2(\nu^2-d^2)}{(\nu^2-1)(\nu^2-a^2)(\nu^2-b^2)(\nu^2-c^2)}.$$

We are looking for an expansion as it follows:

$$P(\nu) = \frac{A_1}{\nu-1} + \frac{A_2}{\nu+1} + \frac{A_3}{\nu-a} + \frac{A_4}{\nu+a} + \frac{A_5}{\nu-b} + \frac{A_6}{\nu+b} + \frac{A_7}{\nu-c} + \frac{A_8}{\nu+c}$$

(22)

For the coefficients $A_k, k \in \{1,2,3,4,5,6,7,8\}$ which occur in the relation (22) we find the following values:

$$\begin{cases} A_1 = -A_2 = \dfrac{1-d^2}{2(1-a^2)(1-b^2)(1-c^2)} \overset{note}{=} \tau_1 \\ A_3 = -A_4 = \dfrac{a(a^2-d^2)}{2(a^2-1)(a^2-b^2)(a^2-c^2)} \overset{note}{=} \tau_2 \\ A_5 = -A_6 = \dfrac{b(b^2-d^2)}{2(b^2-1)(b^2-a^2)(b^2-c^2)} \overset{note}{=} \tau_3 \\ A_7 = -A_8 = \dfrac{c(c^2-d^2)}{2(c^2-1)(c^2-a^2)(c^2-b^2)} \overset{note}{=} \tau_4 \end{cases} \quad (23)$$

If we note $\mu = \frac{2\sigma q(1-\overline{q}\sigma)(1-a^2)^2\sqrt{t_4}}{(\sigma - z)(1-\overline{z}\sigma)}$, for I_2 we obtain the expression :

$$I_2 = \mu\left(\tau_1\ln\frac{\nu-1}{\nu+1} + \tau_2\ln\frac{\nu-a}{\nu+a} + \tau_3\ln\frac{\nu-b}{\nu+b} + \tau_4\ln\frac{\nu-c}{\nu+c}\right) \quad (24)$$

From relation (17) we note that

$$\nu(\zeta) = \sqrt{\frac{\zeta - \frac{t_0}{\rho}\overline{q}}{\zeta - \frac{\rho}{t_4}\overline{q}}} \quad (25)$$

and from the notation made for the computation of I_2 ,

$$u^2(\zeta) = w(\zeta) + a^2 \quad (26)$$

By using the relations (16) and (24) relation (15) becomes:

$$I_1\Big|_0^w = I_2\Big|_0^\zeta$$

For ζ=0 (16),(24) and (15′) we obtain the constant which results from the two factors of relations (16) and (24) (corresponding $\frac{u-\alpha}{u+\alpha}$ and $\frac{\nu-a}{\nu+a}$) :

$$\ln(-1)\,\frac{\alpha\sqrt{f\overline{f}}(-2pl^2-f)}{f} + \mu\tau_2$$

Thus (15′) will be written as it follows:

$$\left\{\begin{array}{l} \frac{\sqrt{f\bar{f}(-2pl^2-f)}}{f}\ln\left[\left(\frac{u(\zeta)-\alpha}{u(\zeta)+\alpha}\right)^{\alpha}\left(\frac{u(\zeta)+\beta}{u(\zeta)-\beta}\right)^{\beta}\right]+\frac{\sqrt{f\bar{f}(-2pl^2-f)}}{f}\ln\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^{\beta}+ \\ +\ln(-1)^{\frac{\alpha\sqrt{f\bar{f}(-2pl^2-f)}}{f}+\mu\tau_2}= \\ =\mu\left[\ln\left(\frac{v(\zeta)-1}{v(\zeta)+1}\right)^{\tau_1}+\ln\left(\frac{v(\zeta)-a}{v(\zeta)+a}\right)^{\tau_2}+\ln\left(\frac{v(\zeta)-b}{v(\zeta)+b}\right)^{\tau_3}+\ln\left(\frac{v(\zeta)-c}{v(\zeta)+c}\right)^{\tau_4}\right]- \\ -\mu\ln\left[\left(\frac{a-1}{a+1}\right)^{\tau_1}\left(\frac{a-b}{a+b}\right)^{\tau_3}\left(\frac{a-c}{a+c}\right)^{\tau_4}\right] \end{array}\right.$$

After restrictions and convenient grouping the equation above is written as it follows :

$$\left\{\begin{array}{l} \left[\left(\frac{u(\zeta)-\alpha}{u(\zeta)+\alpha}\right)^{\alpha}\left(\frac{u(\zeta)+\beta}{u(\zeta)-\beta}\,\frac{\alpha+\beta}{\alpha-\beta}\right)^{\beta}\right]^{\frac{\sqrt{f\bar{f}(-2pl^2-f)}}{f}}\times \\ \times(-1)^{\frac{\alpha\sqrt{f\bar{f}(-2pl^2-f)}}{f}+\mu\tau_2}= \\ =\left[\begin{array}{l}\left(\frac{v(\zeta)-1}{v(\zeta)+1}\,\frac{a+1}{a-1}\right)^{\tau_1}\left(\frac{v(\zeta)-a}{v(\zeta)+a}\right)^{\tau_2}\times \\ \times\left(\frac{v(\zeta)-b}{v(\zeta)+b}\,\frac{a+b}{a-b}\right)^{\tau_3}\left(\frac{v(\zeta)-c}{v(\zeta)+c}\,\frac{a+c}{a-c}\right)^{\tau_4}\end{array}\right]^{\mu} \end{array}\right. \quad (27)$$

Relation (27) implicitly represents the equation verified by the extremal function w=w(ζ) which performs $\max\limits_{f\in S}|f(z)|$.

2°.Let us now assume that $Im(z\bar{l}m)=0$.. In this case from the expressin of p there must be $zl-\bar{z}\bar{l}=0$..From conditions $z\bar{l}m-\bar{z}l\bar{m}=0,\ zl-\bar{z}\bar{l}=0,\ |f'(z)|=1$ and |z|=r we obtain $m=\bar{m}$ and from the expression of p it results that p is complex, which is a contradiction! Therefore this case cannot happen.

5. Next we will show how θ and k can be determined. In relation (14) we perform ζ→z ; and we obtain:

$$-pz^2l^2\bar{f}(1-r^2)^2=(1-\bar{q}z)^2(t_0-2kq\zeta+q^2t_4\zeta^2)$$

and by multiplying the resulting equality by $\bar{z}^2\bar{l}^2f$ we obtain (|z|=r) :

$$-pz^2\bar{z}^2l^2\bar{l}^2f\bar{f}(1-r^2)^2=(\bar{z}-\bar{q}r^2)^2(t_0-2kq\zeta+q^2t_4\zeta^2)f\bar{l}^2 \quad (28)$$

Since the expression on the left of the equality (28) is real, we obtain a system with two equations from which we obtain θ and k :

$$\left\{\begin{array}{l} \mathrm{Re}\left[(\bar{z}-\bar{q}r^2)^2(t_0-2kq\zeta+q^2t_4\zeta^2)f\bar{l}^2\right]+ \\ +pz^2\bar{z}^2l^2\bar{l}^2f\bar{f}(1-r^2)^2=0 \\ \text{and} \\ \mathrm{Im}\left[(\bar{z}-\bar{q}r^2)^2(t_0-2kq\zeta+q^2t_4\zeta^2)f\bar{l}^2\right]=0 \end{array}\right. \quad (29)$$

With θ and k determined in this way the extremal function w = w(ζ) from equation (27) is well determined; with its help we find $\max\limits_{f\in S}|f(z)|$ in conditions

$|f'(z)|=1,\ |z|=r,\ f\in\mathbf{S},\ 0\le r<1,\ r$ given.

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