LOCALLY DIAGONALLY TOPOLOGY OF A GROUPOID G TRANSPORTED TO THE PRINCIPAL GROUPOID ASSOCIATED WITH G

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ABSTRACT : The purpose of this paper is to prove that if each point of a topological groupoid G has a neighborhood basis of diagonally compact sets, then the same is true for the principal groupoid R of G endowed with the transported topology from G, as well as for G endowed with the modified topology with respect to R introduced in [3].

KEY WORDS : topological groupoid; principal groupoid; locally diagonally compact set.

1. INTRODUCTION

In order to define convolution that gives the algebra structure on a function space associated with a topological groupoid one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". By analogy with the group case, it is usual to endow the groupoid with a locally compact topology [6,5]. But unlike the case of locally compact group, Haar system on groupoid need not exists. However in lots of cases one can construct a system of positive Radon measures supported on r-fibres that satisfies the "left invariance" condition but not the "continuity" condition. Moreover the notion of left invariant system (meaning system of positive Radon measures supported on r-fibres that satisfies the "left invariance" condition) has sense on a topological groupoid with locally compact r-fibres. In [4] we proved that the "left invariance" condition it is enough to construct various convolution algebra on a topological groupoid which is T_1, has Hausdorff fibres and it is endowed with a family $\mathcal{K}$ of subsets $K\subset G$ satisfying the following conditions:

1. For every $K\in\mathcal{K}$, $K^{-1}\in\mathcal{K}$.
2. For every $K_1,K_2\in\mathcal{K}$ there is $K_3\in\mathcal{K}$ such that $K_1K_2\subset K_3$.
3. For every $K_1,K_2\in\mathcal{K}$, $K_1\cup K_2\in\mathcal{K}$.
4. For every $u\in G^{(0)}$ and every $K\in\mathcal{K}$, $K\cap G^u$ and $K\cap G_u$ are compact
5. Each point $x\in G$ has a neighborhood basis of sets K belonging to $\mathcal{K}$.

Obviously a locally compact groupoid having Hausdorff unit space endowed with the family of compact subsets satisfies the previously conditions (and consequently, even if the groupoid does not admit a Haar system, various C*-algebras can be associated with it [2,4]).

Also the family of locally diagonally compact subsets (see Definition 3.2) of a topological groupoid G satisfies the conditions 1-4.

In [3] we modified the topology of a groupoid G with locally compact fibres without altering the topology of the fibres and we proved that we can endow the principal groupoid R associated to G with a similar topology. The purpose of this paper is to prove that if each point $x\in G$ has a neighborhood basis of diagonally compact sets (Definition 3.2), then the same is true for the principal groupoid R as well as for G endowed with the modified topology.

2. NOTATION AND TERMINOLOGY

We begin this section by recalling basic definitions, notation and terminology associated with groupoids. A groupoid is like a group with multiplication only partially defined. More accurately, a groupoid is a set G, together with a distinguished subset $G^{(2)} \subset G \times G$, and two maps:
a product map

$$(x, y) \to xy \ [:G^{(2)} \to G],$$

and an inverse map

$$x \to x^{-1} \ [:G \to G],$$

such that the following conditions are satisfied:

1. $(x^{-1})^{-1} = x$
2. If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then (xy, z), $(x,yz) \in G^{(2)}$ and $(xy)z= x(yz)$.
3. $(x, x^{-1}) \in G^{(2)}$, and if $(y, x) \in G^{(2)}$, then $y = (yx)x^{-1}$.
4. $(x^{-1}, x) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $y = x^{-1}(xy)$.

The maps r and d on G, defined by $r(x) = xx^{-1}$ and respectively, $d(x) = x^{-1}x$, are called the range and the domain maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. For every $u \in G^{(0)}$, the fibres of the range and the domain maps over u are denoted $G^u = r^{-1}(\{u\})$ and $G_u = d^{-1}(\{u\})$, respectively. For u and v in $G^{(0)}$, (r,d)-fibre is $G^u_v = G^u \cap G_v$. It is easy to see that G^u_u is a group, called the isotropy group at u. The group bundle

$$\{x \in G: r(x)=d(x)\}$$

is denoted G', and is called the isotropy group bundle of G.

The relation u~v if and only if there is $x \in G$ such that $r(x)=u$ and $d(x)=v$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits. The graph of this equivalence relation

$$R=\{(u,v) \in G^{(0)} \times G^{(0)} : \text{there is } x \in G \text{ such that } r(x)=u \text{ and } d(x)=v\}$$

can be regarded as a groupoid, under the product and inverse maps:

$$(u,v)(v,w) = (u,w)$$
$$(u,v)^{-1} = (v,u)$$

R is called the principal groupoid associated with G. We denote by $(r,d):G \to R$, the map defined by

$$(r,d)(x)=(r(x),d(x)) \text{ for all } x \in G.$$

A topological groupoid (G,τ) consists of a groupoid G and a topology τ compatible with the groupoid structure i.e. the inverse and product map are continuous maps (the topology on $G^{(2)}$ is induced from $G \times G$ endowed with the product topology). Sometimes we will refer to a topological groupoid (G, τ) by just the underlying groupoid G, when it is irrelevant or clear from the context which topology on G is considered.

In [3] we started with a topological groupoid (G, τ_G), we introduced a topology $\tau_R(\tau_G)$ on the principal groupoid R associated with G (called transported topology from G) and a new topology $\tau_{G\vee R}$ on G (called the modified topology on G with respect to R) such that:

1. $(R, \tau_R(\tau_G))$ is a topological groupoid and $\tau_R(\tau_G)$ is finer that the quotient topology on R.
2. $(G, \tau_{G\vee R})$ is a topological groupoid and $\tau_{G\vee R}$ is finer than τ_G.
3. For every $u \in G^{(0)}$, $\tau_{G\vee R}$ and τ_G induce the same subspace topologies on G^u and G_u.
4. If $(r,d):(G, \tau_G) \to R$ is open with respect to the quotient topology on R, then $\tau_{G\vee R} = \tau_G$.
5. The map $(r,d):(G, \tau_{G\vee R}) \to (R, \tau_R(\tau_G))$ is an open continuous map. Consequently,

   5.a) the quotient topology on R with respect to this map, $\tau_R(\tau_G)$ and $\tau_R(\tau_{G\vee R})$ coincide.

   5.b) the subspace topology on $G^{(0)}$ with respect to $\tau_{G\vee R}$ is the same with the topology on $G^{(0)}$ viewed as a unit space of R (endowed with $\tau_R(\tau_G)=\tau_R(\tau_{G\vee R})$).

   5.c) if $r':G' \to G^{(0)}$, $r'(x)=r(x)$ is the restriction of the range map to G', isotropy group bundle of G, then r' is an open map, when G' and $G^{(0)}$ are endowed with the subspace topologies from G with respect to $\tau_{G\vee R}$.

   5.d) the range map of G endowed with $\tau_{G\vee R}$ is open if and only if the range map of $(R, \tau_R(\tau_G))$ is open.

Let us recall that a basis for the topology $\tau_R(\tau_G)$ is given by the family of sets $\{U(F)\}_F$, where each F is a finite collection F of open subsets of G (i.e. $F \subset \tau_G$) and

$$U(F) = \bigcap_{U \in F} (r,d)(U)$$

Moreover a basis for the topology $\tau_{G \vee R}$ is given by

$$V \bigcap \left(\bigcap_{U \in F} (r,d)^{-1}((r,d)(U)) \right)$$

where V runs over all open sets of G and F runs over all finite collections of open subsets of G.

3. LOCALLY DIAGONALLY COMPACT GROUPOIDS

Definition 3.1. A subset K of a topological groupoid G is called diagonally compact if the following conditions are satisfied:

1. Every net $(x_i)_i$ in K such that $(r(x_i))_i$ converges in $G^{(0)}$ has a convergent subnet to an element $x \in K$.
2. Every net $(x_i)_i$ in K such that $(d(x_i))_i$ converges in $G^{(0)}$ has a convergent subnet to an element $x \in K$.

It is easy to see that if G is a topological groupoid then:

1. For every diagonally compact set $K \subset G$, K^{-1} is diagonally compact.
2. For every diagonally compact sets $K_1, K_2 \subset G$, K_1K_2 is diagonally compact.
3. For every diagonally compact sets $K_1, K_2 \subset G$, $K_1 \cup K_2$ is diagonally compact.
4. For every $u \in G^{(0)}$ and every diagonally compact set $K \subset G$, $K \cap G^u$ and $K \cap G_u$ are compact.
5. If G is Hausdorff and K is a diagonally compact subset, then K is closed.
6. If the topology induced on the unit space $G^{(0)}$ is Hausdorff and K is a diagonally compact subset of G, then r(K) and d(K) are closed as subsets of $G^{(0)}$.
7. If the topology induced on the unit space $G^{(0)}$ is Hausdorff and K is a diagonally compact subset of G, then $(r,d)^{-1}((r,d)(K))$ is closed as subsets of G or equivalently, (r,d)(K) is closed as a subset of the principal groupoid R with respect to the quotient topology induced by the map $(r,d): G \rightarrow R$.
8. If $K \subset G$ is diagonally compact and $F \subset G$ is closed, then $K \cap F$ is diagonally compact
9. If $K \subset G$ is diagonally compact and $F \subset G$ is closed, then KF and FK are closed.
10. If topology induced on $G^{(0)}$ is locally compact then the above notion of diagonally compact coincide with that used in [5] and with the notion of conditionally compact used in [7] (K is diagonally compact iff $K \cap r^{-1}(L)$ and $K \cap d^{-1}(L)$ are compact whenever L is a compact subset of $G^{(0)}$).

Definition 3.2. Let G be a topological groupoid such that:

1. The points are closed in G (or equivalently, G is a T_1-space).
2. $G^{(0)}$ is a Hausdorff subspace of G.

Then G is said to be a locally diagonally compact groupoid if each point $x \in G$ has a neighborhood basis of diagonally compact sets.

Let us recall that if G is a topological groupoid with the property that the points are closed in G and if $G^{(0)}$ is a Hausdorff subspace of G, then for every $u \in G^{(0)}$, G^u and G_u are Hausdorff (see Proposition 2.8/p. 569 [8]).

Proposition 3.3. If (G, τ_G) is a topological groupoid with Hausdorff unit space and K is a diagonally compact subset of G with respect to τ_G, then (r,d)(K) is a diagonally compact subset of the principal groupoid R associated with G endowed with $\tau_R(\tau_G)$.

Proof. We shall use the fact (proved in [3]) that a net $(u_i,v_i)_i$ converges to (u,v) in R iff for each $x \in G_v^u$ every subnet of $(u_i,v_i)_i$ has a subnet, also denoted $((u_i,v_i))_i$, with the property that there are x_i such that

$(r,d)(x_i) = (u_i,v_i)$ for all i

$(x_i)_i$ converges to x.

Let $(u_i,v_i)_i$ be net in K such that $(u_i)_i$ converges with respect to the topology induced on $G^{(0)}$ by $\tau_R(\tau_G)$. Then there is a net $(x_i)_i$ such that

$x_i \in K$

$(u_i,v_i) = (r(x_i), d(x_i))$ for all i

The fact that $\tau_R(\tau_G)$ induced on $G^{(0)}$ a topology finer than τ_G, implies that $(r(x_i))_i = (u_i)_i$ converges also in the topology induced by τ_G. Since K is diagonally compact with respect to τ_G, it follows that there is a subnet of $(x_i)_i$, also denoted $(x_i)_i$, that converges with respect to τ_G to an element $x\in K$. The fact that unit space $G^{(0)}$ is Hausdorff implies that the limit of $(u_i)_i$ is r(x). Let y be such that (r,d)(y)=(r,d)(x). Then $yx^{-1}\in G'$ and $(u_i)_i$ converges to $r(yx^{-1})=r(x)$ with respect to $\tau_R(\tau_G)$. Consequently, every subnet of $(u_i)_i$ has a subnet, also denoted $(u_i)_i$, with the property that there is net $(z_i)_i$ such that:

$(r,d)(z_i) =(u_i,u_i)$ for all i

$(z_i)_i$ converges to yx^{-1} (with respect to τ_G).

Then

$(r,d)(z_ix_i) =(r(x_i),d(x_i))=(u_i,v_i)$ for all i

$(z_ix_i)_i$ converges to $yx^{-1}x=y$ (with respect to τ_G).

Hence passing to a subnet $(u_i,v_i)_i$ converges to $(r(x),d(x))\in(r,d)(K)$.

Similarly, we can prove that if $(u_i,v_i)_i$ is net in K such that $(v_i)_i$ converges with respect to the topology induced on $G^{(0)}$ by $\tau_R(\tau_G)$, then $(u_i,v_i)_i$ has a subnet converging to an element $(u,v)\in(r,d)(K)$.

Proposition 3.4. If (G,τ_G) is a locally diagonally compact groupoid and if R is the principal groupoid associated with G, then R endowed with the transported topology from G $\tau_R(\tau_G)$ is a locally diagonally compact groupoid.

Proof. Since $G^{(0)}$ is Hausdorff, it follows that R is Hausdorff. Let $(u,v)\in R$ and

$$U(F) = \bigcap_{U\in F}(r,d)(U)$$

be an open neighborhood of (u,v) with respect to $\tau_R(\tau_G)$,where F is a finite collection F of open subsets of G (i.e. $F \subset \tau_G$). For each $U\in F$, let $x_U\in U$ be such that $(u,v)=(r,d)(x_U)$ and let K_U be a diagonally compact neighborhood of x_U with respect to τ_G such that $K_U\subset U$. Then $(r,d)(K_U)$ is closed with respect to the quotient topology on R and consequently, with respect to $\tau_R(\tau_G)$ which is finer than quotient topology [3].

Let $U_0\in F$ and let us denote $L_0=(r,d)(K_{U_0})$. Then L_0 is a diagonally compact neighborhood of (u,v) with respect to $\tau_R(\tau_G)$ and also since $\bigcap_{U\in F\setminus\{U_0\}}(r,d)(K_U)$ is closed, it follows that

$$L_0\bigcap\left(\bigcap_{U\in F\setminus\{U_0\}}(r,d)(K_U)\right)$$

is a diagonally compact neighborhood of (u,v) with respect to $\tau_R(\tau_G)$. Moreover

$$L_0\bigcap\left(\bigcap_{U\in F\setminus\{U_0\}}(r,d)(K_U)\right)\subset\bigcap_{U\in F}(r,d)(U).$$

Proposition 3.5. If (G,τ_G) is a topological groupoid with Hausdorff unit space and K is a diagonally compact subset of G with respect to τ_G, then K is also diagonally with respect to $\tau_{G\vee R}$.

Proof. We shall use the fact [3] the a net $(x_i)_i$ converges to x with respect to $\tau_{G\vee R}$ iff

$(x_i)_i$ converges to x with respect to τ_G

$((r(x_i),d(x_i))_i$ converges to $((r(x),d(x))$ with respect to $\tau_R(\tau_G)$

Let $(x_i)_i$ be a net in K such that $(r(x_i))_i$ converges with respect to the topology on $G^{(0)}$ induced by $\tau_{G\vee R}$. Since $\tau_{G\vee R}$ is finer than τ_G, it follows that $(r(x_i))_i$ also converges with respect to τ_G. The fact that K is diagonally compact with respect to τ_G, implies that there is a subnet of $(x_i)_i$, also denoted $(x_i)_i$, that converges with respect to τ_G to an element $x\in K$. The unit space $G^{(0)}$ being Hausdorff, the limit of $(r(x_i))_i$ must be r(x). Let y be such that (r,d)(y)=(r,d)(x). Then $yx^{-1}\in G'$ and $(r(x_i))_i$ converges to r(y)=r(x) with respect to $\tau_{G\vee R}$. Consequently, every subnet of $(r(x_i))_i$ has a subnet, also denoted $(r(x_i))_i$, with the property that there is net $(z_i)_i$ such that:

$(r,d)(z_i) =(r(x_i),r(x_i))$ for all i

$(z_i)_i$ converges to yx^{-1} (with respect to τ_G).

Then

$(r,d)(z_ix_i) =(r(x_i),d(x_i))$ for all i

$(z_ix_i)_i$ converges to $yx^{-1}x=y$ (with respect to τ_G).

Hence $(x_i)_i$ has a subnet that converges to x with respect to $\tau_{G\vee R}$.

Similarly, if $(x_i)_i$ is a net in K such that $(d(x_i))_i$ converges with respect to the topology on $G^{(0)}$ induced by $\tau_{G\vee R}$, then $(x_i)_i$ has a net converging to an element $x\in K$.

Proposition 3.6. If (G,τ_G) is a locally diagonally compact groupoid, then $(G, \tau_{G\vee R})$ is a locally diagonally compact groupoid.

Proof. Let $x\in G$ and

$$V\bigcap\left(\bigcap_{U\in F}(r,d)^{-1}((r,d)(U))\right)$$

be an open neighborhood of x with respect to $\tau_{G\vee R}$, where $V\in\tau_G$ and F is finite collection of open subsets of G ($F\subset\tau_G$). For each $U\in F$, let $x_U\in U$ be such that $(r,d)(x)=(r,d)(x_U)$ and let K_U be a diagonally compact neighborhood of x_U with respect to τ_G such that $K_U\subset U$. Then $(r,d)^{-1}((r,d)(K_U))$ is closed with respect to τ_G which is coarser that $\tau_{G\vee R}$ [3]. If K is a diagonally compact neighborhood of x with respect to τ_G such that $K\subset V$, then K is a diagonally compact neighborhood of x with respect to $\tau_{G\vee R}$ and also

$$K\bigcap\left(\bigcap_{U\in F}(r,d)^{-1}((r,d)(K_U))\right)$$

is a diagonally compact neighborhood of x with respect to $\tau_{G\vee R}$ (because

$$\bigcap_{U\in F}(r,d)^{-1}((r,d)(K_U))$$

is closed. Moreover

$$K\bigcap\left(\bigcap_{U\in F}(r,d)^{-1}((r,d)(K_U))\right)$$

is included in

$$V\bigcap\left(\bigcap_{U\in F}(r,d)^{-1}((r,d)(U))\right).$$

Proposition 3.7. If (G,τ_G) is a topological groupoid with open range map, then the principal groupoid R associated with G endowed with $\tau_R(\tau_G)$ has open range map.

Proof. We shall use the following fact: The surjective map $p:X\to Y$ is an open map if and only if for every $x\in X$, given a net $(y_i)_i$ converging to p(x) in Y, there is a subnet $(y_{i_j})_j$ of $(y_i)_i$ and a net $(x_j)_j$ indexed by the same set such that $(x_j)_j$ converges to x and $p(x_j)=y_{i_j}$ for all j.

Let $(u_i)_i$ be a net converging to u with respect to the topology on $G^{(0)}$ induced by $\tau_R(\tau_G)$ and let v be such that $(u,v)\in R$. Let $x\in G$ be such that $(r,d)(x)=(u,v)$. Since the range map of (G,τ_G) is open, it follows that there is a net $(x_i)_i$ converging to x (eventually passing to a subnet) with respect to τ_G and having the property that $r(x_i)=u_i$ for all i. Let us prove that $((u_i,d(x_i)))_i$ converges to (u,v). Let $y\in G$ be such that $(r,d)(y)=(u,v)$. Then $yx^{-1}\in G'$, $r(yx^{-1})=u$ and $(u_i)_i$ converges to u with respect to the topology on $G^{(0)}$ induced by $\tau_R(\tau_G)$. Hence there is a net $(z_i)_i$ converging to yx^{-1} (eventually passing to a subnet) with respect to τ_G and having the property that $r(z_i)=u_i$ for all i. Thus $(z_ix_i)_i$ converges to $yx^{-1}x=y$ and $(r,d)(z_ix_i)=(u_i,d(x_i))$ for all i. Consequently, $((u_i,d(x_i)))_i$ converges to (u,v) with respect to $\tau_R(\tau_G)$. Therefore the range map of R is open.

If (G,τ_G) is a topological Hausdorff groupoid and K is a locally diagonally compact subset of G, then K is closed. Thus each locally diagonally compact Hausdorff groupoid is regular in the sense that or any point $x\in G$ and neighborhood V of x, there is a closed neighborhood F of x that is a subset of V. However if G is not Hausdorff this is no longer true.

Proposition 3.8. If (G,τ_G) is a locally diagonally compact groupoid and if R is the principal groupoid associated with G, then the unit space of R endowed with the transported topology from G $\tau_R(\tau_G)$ is regular.

Proof. Let $u\in G^{(0)}$ and let

$$\bigcap_{U\in F}r(U\cap G')$$

be neighborhood of u. For each $U\in F$, let $x_U\in U\cap G'$ be such that $u=r(x_U)$ and let K_U be a diagonally compact neighborhood of x_U with respect to τ_G such that $K_U\subset U$. Since $G^{(0)}$ is Hausdorff, it follows that G' is closed. Hence $K_U\cap G'$ is diagonally compact and consequently, $r(K_U\cap G')$ is closed with respect to $\tau_R(\tau_G)$ for all U. Since r is an open map with respect to $\tau_R(\tau_G)$, $r(K_U\cap G')$ is a neighborhood of $u=r(x_U)$ for all U. Thus $\bigcap_{U\in F}r(K_U\cap G')$ is a closed neighborhood of u and $\bigcap_{U\in F}r(K_U\cap G')\subset\bigcap_{U\in F}r(U\cap G')$.

Let (G,τ_G) be a locally diagonally compact groupoid endowed with a left invariant system of measures (pre-Haar system [4]) and $\mathcal{UF}_{dc}(G)$ be the space of functions f:G→**C** which are left and right "uniformly continuous on fibres" (in the sense of [1,2,4]) and which vanish outside a diagonally compact set. Then $\mathcal{UF}_{dc}(G)$ is a *-algebra with respect to the usual involution and convolution [4]. If W is a diagonally compact neighborhood of the unit space $G^{(0)}$, K a diagonally compact subset of G and f: G→**C** is function with the properties that:

1. f vanishes outside K
2. f is continuous on WWKWW

then f is left and right "uniformly continuous on fibres" (we can apply Proposition 2.3/p. 163 [3] with K_0=WKW).

REFERENCES

[1]Buneci, M., Haar systems and homomorphism on groupoids, Operator algebras and mathematical physics (Constanta, 2001), 35--49, Theta, Bucharest, 2003.

[2]Buneci, M. Uniformly continuous functions on non-Hausdorff groupoids, Surveys in Mathematics and its Applications, 5 (2010), 229-245.

[3]Buneci, M. Topological groupoids with locally compact fibres, Topology Proceedings 37 (2011), 239-258.

[4]Buneci, M. Convolution algebras for topological groupoids with locally compact fibres, Opuscula Mathematica 31 (2011), no. 2, 159-172.

[5]Muhly P. and Williams D., Renault's Equivalence Theorem for groupoid crossed products, NYJM Monographs 3, 2008.

[6]Renault J. A groupoid approach to C*-algebras, Lecture Notes in Math., Springer-Verlag, 793, 1980.

[7]Renault J. The ideal structure of groupoid crossed product algebras, J. Operator Theory, 25 (1991), 3-36.

[8]Tu, J.L. Non-Hausdorff groupoids, proper Actions and K-theory, Documenta Math. 9 (2004), 565-597.