TOPOLOGIES DEFINED IN TERMS OF PRE-HAAR SYSTEMS

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Abstract: *We start with a locally diagonally compact groupoid* (G,τ_G) *in the sense of [4], a pre-Haar system* ν *on G and a family A of left and right uniformly continuous on fibres functions on G. The purpose of this short note is to introduce a new locally diagonally compact topology* $\tau(\tau_G,A,\nu)$ *on G such that* τ_G *and* $\tau(\tau_G,A,\nu)$ *induce the same subspace topology on the groupoid fibres and the pre-Haar system* ν *becomes continuous for the functions in A.*

Key words: *topological groupoid; locally diagonally compact set; pre-Haar system.*

1. NOTATION AND TERMINOLOGY

The motivation of this paper is the construction of convolution algebras associated to a groupoid. We shall use the same notation as in [4]. Let us recall that in [4] by a diagonally compact set K we meant a subset of a topological groupoid G satisfying the following conditions:

1. Every net $(x_i)_i$ in K such that $(r(x_i))_i$ converges in $G^{(0)}$ has a convergent subnet to an element $x\in K$.
2. Every net $(x_i)_i$ in K such that $(d(x_i))_i$ converges in $G^{(0)}$ has a convergent subnet to an element $x\in K$.

It is obviously, that if the topology induced on $G^{(0)}$ is locally compact, then the above notion of diagonally compact coincides with that used in [5] and with the notion of conditionally compact used in [7] (K is diagonally compact iff $K\cap r^{-1}(L)$ and $K\cap d^{-1}(L)$ are compact whenever L is a compact subset of $G^{(0)}$). In this paper we shall use a slightly more general definition of a locally diagonally compact groupoid than in [4]:

Definition 1.1. Let G be a topological groupoid such that:

1. The points are closed in G (or equivalently, G is a T_1-space).
2. For all $u\in G^{(0)}$, G^u and G_u are Hausdorff subspaces of G.

Then G is said to be a locally diagonally compact groupoid if each point $x\in G$ has a neighborhood basis of diagonally compact sets. Let us also recall the definition of left and right uniformly continuous on fibres functions on G [1,2,3]. Let G be a topological groupoid and E be a Banach space. The function $f:G\to E$ is said to be left uniformly continuous on fibres if and only if for each $\varepsilon>0$ there is a neighborhood W_ε of the unit space $G^{(0)}$ such that:

$$\|f(sy)-f(y)\|<\varepsilon \text{ for all } s\in W_\varepsilon \text{ and } y\in G^{d(s)}.$$

The function $f:G\to E$ is said to be right uniformly continuous on fibres if and only if for each $\varepsilon>0$ there is a neighborhood W_ε of the unit space $G^{(0)}$ such that:

$$\|f(ys)-f(y)\|<\varepsilon \text{ for all } s\in W_\varepsilon \text{ and } y\in G_{r(s)}.$$

Let us denote by :

1. UF(G) the space of complex valued functions on G which are left and right uniformly continuous on fibres.
2. $UF_{dc}(G)$ consisting in functions $f:G\to\mathbf{C}$ which vanish outside a diagonally compact set K.

Let us remark that if G is a locally diagonally compact groupoid, then for all $u\in G^{(0)}$, G^u and G_u are in fact locally compact Hausdorff subspaces of G.

Indeed, each point x in G^u (respectively, in G_u) has a diagonally compact neighborhood K in G. Then $K\cap G^u$ (respectively, $K\cap G_u$) is a compact neighborhood of x in G^u (respectively, in G_u).

Definition 1.2. Let G be a locally diagonally compact groupoid. A (left) pre-Haar system on G is a family of positive measures,

$$\nu=\{\nu^u,\ u\in G^{(0)}\},$$

with the following properties:

1. ν^u is a positive Radon measure on G^u for all $u\in G^{(0)}$.
2. $\int f(y)d\nu^{r(x)}(y)= \int f(xy)d\nu^{d(x)}(y)$ for all $x\in G$ and all $f\in UF_{dc}(G)$.

The pre-Haar system $\nu=\{\nu^u, u\in G^{(0)}\}$ on G is said to be bounded on compact sets if

$$\sup\{\nu^{u_i}(K),\ i\in I\} < \infty$$

for all nets $(u_i)_{i\in I}$ which converge in $G^{(0)}$ and all diagonally compact subsets K of G.

Notation 1.3 Let G be a locally diagonally compact groupoid and $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G. For each $f\in UF_{dc}(G)$, let us denote φ_f the function defined by

$$\varphi_f(u)= \int f(y)d\nu^u(y) \text{ for all } u\in G^{(0)}.$$

For each $f\in UF(G)$, let us denote by

$$||f||_{I,r}=\sup\{\int|f(y)|d\nu^u(y),\ u\in G^{(0)}\}$$

$$||f||_{I,d}=\sup\{\int|f(y^{-1})|d\nu^u(y),\ u\in G^{(0)}\}$$

$$||f||_I=\max\{||f||_{I,r},\ ||f||_{I,d}\}$$

and let us consider

$$UF_I(G)=\{f\in UF(G):||f||_I<\infty\} .$$

For $f\in UF(G)$, the involution is defined by

$$f^*(x)= \overline{f}(x^{-1}),\ x\in G.$$

For $f,g\in UF_I(G)$ (or $f,g\in UF_{dc}(G)$) the convolution is defined by:

$$f*g(x)= \int f(y)g(y^{-1}x)d\nu^{r(x)}(y),\ x\in G.$$

By Theorem 3.4/p. 167 [3], $UF_I(G)$ and $UF_{dc}(G)$ are *-algebras.

2. TOPOLOGIES DEFINED IN TERMS OF PRE-HAAR SYSTEMS

Proposition 2.1. Let G be a locally diagonally compact groupoid, let $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G and $A\subset UF_{dc}(G)$. Let us denote by $\Phi(A,\nu)$ the family of subsets U of $G\times G$ with the property that there are a positive ε and a finite subset F of A such that U contains the intersection of

$$\{(x,y): |\varphi_f(d(x))- \varphi_f(d(y))|<\varepsilon \text{ for all } f\in F\}$$

and

$$\{(x,y): |\varphi_f(r(x))- \varphi_f(r(y))|<\varepsilon \text{ for all } f\in F\}.$$

Then $\Phi(A,\nu)$ is a uniform structure on G.

Proof. For a positive ε and a finite subset F of A let us denote by $U(\varepsilon,F)$ the intersection of

$$\{(x,y): |\varphi_f(d(x))- \varphi_f(d(y))|<\varepsilon \text{ for all } f\in F\}$$

and

$$\{(x,y): |\varphi_f(r(x))- \varphi_f(r(y))|<\varepsilon \text{ for all } f\in F\}.$$

It is easy to check that

1. If $U\in\Phi(A,\nu)$, then $\{(x,x):x\in G\}\subset U$.
2. If $U\in\Phi(A,\nu)$ and $U\subset V\subset G\times G$, then $V\in\Phi(A,\nu)$.
3. If $U\in\Phi(A,\nu)$ and $V\in\Phi(A,\nu)$, then
$$U\cap V\in\Phi(A,\nu),$$
since $U(\min(\varepsilon_1,\varepsilon_2),F_1\cup F_2)$ is contained in $U(\varepsilon_1,F_1)\cap U(\varepsilon_2,F_2)$.
4. If $U\in\Phi(A,\nu)$, then there is $V\in\Phi(A,\nu)$ such that $VV\subset U$, where
$$VV=\{(x,z):\exists y\in X:(x,y)\in V \text{ and } (y,z)\in V\}.$$
Indeed, $U(\varepsilon/2,F)\ U(\varepsilon/2,F)\subset U(\varepsilon,F)$.
5. If $U\in\Phi(A,\nu)$, then
$$U^{-1}=\{(y,x):(x,y)\in U\}=U\in\Phi(A,\nu).$$

Thus $\Phi(A,\nu)$ is a uniform structure (or a uniformity) on G.

Definition 2.2. Let G be a locally diagonally compact groupoid (whose topology will be denoted τ_G), let $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G and $A\subset UF_{dc}(G)$. The uniform structure $\Phi(A,\nu)$ defined in Proposition 2.1 will be called the uniform structure induced by A and $\nu=\{\nu^u, u\in G^{(0)}\}$. The coarsest topology on G which is finer than τ_G and finer than the topology on G induced by the uniform structure $\Phi(A,\nu)$ will be called the topology induced by A and $\nu=\{\nu^u, u\in G^{(0)}\}$ and will be denoted by $\tau(\tau_G,A,\nu)$.

It is easy to see that a subset U is open with respect to $\tau(\tau_G,A,\nu)$ if and only if for each $x\in U$, there are

1. an open set $V\in\tau_G$
2. $\varepsilon>0$
3. a finite subset $F\subset A$

such that

$$x\in V\cap U(\varepsilon,F)\subset U.$$

Proposition 2.3. Let G be a locally diagonally compact groupoid (whose topology will be denoted τ_G), let $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G and $A\subset UF_{dc}(G)$. Then G endowed with $\tau(\tau_G,A,\nu)$ is a topological groupoid.

Proof. For each $x\in G$, let us denote by $U_x(\varepsilon,F)$ the intersection of

$$\{y: |\varphi_f(d(x))- \varphi_f(d(y))|<\varepsilon, \text{ for all } f\in F\}$$

and

$$\{y: |\varphi_f(r(x))- \varphi_f(r(y))|<\varepsilon \text{ for all } f\in F\},$$

where $\varepsilon>0$ and F is a finite subset of A.

If $(x,y)\in G^{(2)}$, then for every $\varepsilon>0$ and finite $F\subset A$, we have

$$U_x(\varepsilon/2,F)\, U_y(\varepsilon/2,F)\subset U_{xy}(\varepsilon,F)$$

and

$$U_x(\varepsilon,F)^{-1}=U_{x^{-1}}(\varepsilon,F).$$

Since in addition (G, τ_G) is a topological groupoid, it follows that G endowed with $\tau(\tau_G,A,\nu)$ is a topological groupoid.

According Lemma 2.10/p. 10 [7], if G is a locally Hausdorff, locally compact groupoid (in the sense of [5] and [7] or [6] in the Hausdorff case) and if $G^{(0)}$ is paracompact, then G has a diagonally compact neighborhood of $G^{(0)}$. In the following we shall assume the existence of such neighborhood of the unit space.

Lema 2.4. Let G be a locally diagonally compact groupoid and let $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G. If there is a diagonally compact neighborhood W_0 of the unit space $G^{(0)}$, then for each $f\in UF_{dc}(G)$ and each $u\in G^{(0)}$, the restriction of $\varphi_f\circ d$ to G^u (as well as the restriction of $\varphi_f\circ r$ to G_u) is continuous.

Proof. Let $(x_i)_i$ be a net in G^u that converges to x. Then $(x_i^{-1}x)_i$ converges to d(x). Since $f\in UF(G)$, it follows that for each $\varepsilon>0$, there is a neighborhood $W_\varepsilon\subset W_0$ of the unit space such that

$$|f(sy)-f(y)| <\varepsilon \text{ for all } s\in W_\varepsilon \text{ and all } y\in G^{d(s)}.$$

Furthermore since $(x_i^{-1}x)_i$ converges to $d(x)\in W_\varepsilon$, it follows that there is i_ε such that for all $i\geq i_\varepsilon$, $x_i^{-1}x\in W_\varepsilon$ and

$$|f(x_i^{-1}xy)-f(y)| <\varepsilon \text{ for all } y\in G^{d(x)}.$$

Thus for all $i\geq i_\varepsilon$,

$$|\varphi_f(d(x_i))- \varphi_f(d(x))|$$
$$=|\int f(x_i^{-1}xy)d\nu^{d(x)}(y)- f(y)d\nu^{d(x)}(y)|$$
$$\leq\int|f(x_i^{-1}xy) - f(y)|d\nu^{d(x)}(y)$$
$$\leq\varepsilon\nu^{d(x)}(W_0^{-1}K\cap G^{d(x)}).$$

Proposition 2.5. Let (G,τ_G) be a locally diagonally compact groupoid having a locally diagonally compact neighborhood of the unit space, let $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G and $A\subset UF_{dc}(G)$. Then τ_G and $\tau(\tau_G,A,\nu)$ induce the same subspace topology on G^u for all $u\in G^{(0)}$.

Proof. For each $x\in G$, each $\varepsilon>0$ and each finite subset F of A, let us denote by $U_x(\varepsilon,F)$ the intersection of

$$\{y: |\varphi_f(d(x))- \varphi_f(d(y))|<\varepsilon, \text{ for all } f\in F\}$$

and

$$\{y: |\varphi_f(r(x))- \varphi_f(r(y))|<\varepsilon \text{ for all } f\in F\}.$$

Since for each $f\in A$ and each $u\in G^{(0)}$, the restriction of $\varphi_f\circ d$ to G^u is a continuous map and the restriction of $\varphi_f\circ r$ to G^u is a constant map, it follows that for every $x\in G^u$, $U_x(\varepsilon,F)\cap G^u$ is open in G^u. Hence τ_G and $\tau(\tau_G,A,\nu)$ induce the same subspace topology on G^u for all $u\in G^{(0)}$.

Proposition 2.6. Let (G,τ_G) be a locally diagonally compact groupoid with the property that the subspace topology induced by τ_G on $G^{(0)}$ is Hausdorff. Let $\nu=\{\nu^u, u\in G^{(0)}\}$ be a pre-Haar system on G bounded on compact sets and $A\subset UF_{dc}(G)$ be such that for each $f\in A$ and each net $(x_i)_i$ converging to x in G, the function $y\rightarrow f(xy)$ can be extended to a function $g\in A$ satisfying the following condition:

$$\lim_i \sup_y |f(x_iy)-g(y)|=0$$

Then any diagonally compact subset of G with respect to τ_G, is also diagonally compact with respect to $\tau(\tau_G,A,\nu)$.

Proof. Let us observe that if $u\in G^{(0)}$, $\varepsilon>0$ and F is a finite subset of A, then

$$U_u(\varepsilon,F)\cap G^{(0)}$$
$$=\{v\in G^{(0)}:|\varphi_f(v)- \varphi_f(u)|<\varepsilon \text{ for all } f\in F\}.$$

Let K be a diagonally compact subset of G with respect to τ_G and let $(x_i)_i$ be a net in K such that $(r(x_i))_i$ converges with respect to the topology on $G^{(0)}$ induced by $\tau(\tau_G,A,\nu)$. Since $\tau(\tau_G,A,\nu)$ is finer than τ_G, it follows that $(r(x_i))_i$ also converges with respect to τ_G. The fact that K is diagonally compact with respect to τ_G, implies that there is a subnet of $(x_i)_i$, also denoted $(x_i)_i$, that converges with respect to τ_G to an element $x\in K$.

The map r being continuous, $(r(x_i))_i$ converges to r(x) with respect to the topology on $G^{(0)}$ induced by τ_G. On the other hand $(r(x_i))_i$ converges to an element u_0 with respect to the topology on $G^{(0)}$ induced by $\tau(\tau_G,A,\nu)$, and consequently with respect to the topology on $G^{(0)}$ induced by τ_G. Since $G^{(0)}$ is Hausdorff, it follows that $u_0=r(x)$. For each $f\in F$, let g_f be a function that extends $y\to f(x^{-1}y)$ and has the property that

$$\lim_i \sup_y |f(x_i^{-1}y)-g_f(y)|=0.$$

Then we have

$$|\varphi_f(d(x_i))- \varphi_f(d(x))|$$
$$=|\int f(x_i^{-1}y)d\nu^{r(x_i)}(y)- f(x^{-1}y)d\nu^{r(x)}(y)|$$
$$\leq\int|f(x_i^{-1}y) - g_f(y)|d\nu^{r(x_i)}(y)+ |\int g_f(y)d\nu^{r(x_i)}(y)- \int g_f(y)d\nu^{r(x)}(y)|$$

Therefore $(d(x_i))_i$ converges to d(x) with respect to the topology on $G^{(0)}$ induced by $\tau(\tau_G,A,\nu)$. Hence $(x_i)_i$ converges to x with respect to τ_G, $(r(x_i))_i$ converges to r(x) and $(d(x_i))_i$ converges to d(x) with respect to $\tau(\tau_G,A,\nu)$. Consequently, $(x_i)_i$ converges to x with respect to $\tau(\tau_G,A,\nu)$.

Similarly, every net $(x_i)_i$ in K such that $(d(x_i))_i$ converges in $G^{(0)}$ with respect to $\tau(\tau_G,A,\nu)$ has a convergent subnet to an element $x\in K$.

Example 2.7. Let (G, τ_G) be a topological groupoid with the following properties:

1. Each point $x\in G$ has a compact Hausdorff neighborhood.
2. The subspace topology induced by τ_G on $G^{(0)}$ is Hausdorff.

(a locally compact groupoid in the sense of [5,7,8]). According Proposition 2.8/p. 569 [8], if G is a topological groupoid with the property that the points are closed in G and if $G^{(0)}$ is a Hausdorff subspace of G, then for every $u\in G^{(0)}$, G^u and G_u are Hausdorff. Thus (G,τ_G) is a locally diagonally grupoid in our sense.

Let A be the family of the functions $f:G\to\mathbf{C}$ which vanish outside a compact set K contained in an open Hausdorff subset U of G and being continuous on U. Then G and A satisfy the hypothesis of Proposition 2.6.

Let us exemplify how the new topology can be used to construct convolution algebras. If the groupoid (G, τ_G), the family A and the pre-Haar system ν satisfy the hypotheses of Proposition 2.5 and Proposition 2.6, then $(G,\tau(\tau_G,A,\nu))$ is a locally diagonally compact groupoid for which ν is also a pre-Haar system. Consequently, $UF_{dc}(G)$ (defined with respect to the new topology $\tau(\tau_G,A,\nu)$) is a *-algebra with respect to the usual involution and convolution (see [3]). We also can take into consideration the *-algebra generated by the functions f such that φ_f are continuous on $G^{(0)}$ (with respect to subspace topology induced by $\tau(\tau_G,A,\nu)$).

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