A GROUPOID ASSOCIATED TO DISCRETE-TIME SYSTEM THAT DOES NOT SATISFIES SEMIGROUP PROPERTY

Mădălina Roxana Buneci, *University Constantin Brâncuşi of Târgu*-Jiu, ROMANIA

ABSTRACT: The purpose of this short note is to introduce a groupoid associated to a function f: X×□ →X where X is a uniform space. The motivation is given by the discrete time systems of the form $x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_0)$ that do not satisfy the semigroup property of a process. For these systems the groupoid is induced by the function f defined by $f(x,n) := f_n(x_{n-1}, x_{n-2}, \ldots, x_0)$ with $x_0 = x$ (or $x_{n_0} = x$ for a fixed n_0). The notions of stability and asymptotic stability are introduced in the groupoid framework.

KEY WORDS: groupoid; equilibrium point; asymptotically stable equilibrium point.

1. INTRODUCTION

A groupoid is a set G, together with a distinguished subset $G^{(2)} \subset G\times G$, and two operations:

product $(\gamma_1, \gamma_2) \to \gamma_1\gamma_2$ [$:G^{(2)} \to G$]

inversion $\gamma \to \gamma^{-1}$ [$:G \to G$]

such that the following relations are satisfied:

1. $(\gamma^{-1})^{-1} = \gamma$
2. If $(\gamma_1,\gamma_2)\in G^{(2)}$ and $(\gamma_2,\gamma_3)\in G^{(2)}$, then $(\gamma_1\gamma_2, \gamma_3)$, $(\gamma_1,\gamma_2\gamma_3) \in G^{(2)}$ and $(\gamma_1\gamma_2)\,\gamma_3 = \gamma_1(\gamma_2, \gamma_3)$.
3. $(\gamma, \gamma^{-1})\in G^{(2)}$, and if $(\gamma_1, \gamma)\in G^{(2)}$, then $(\gamma_1\gamma)\gamma^{-1} = \gamma_1$.
4. $(\gamma^{-1}, \gamma)\in G^{(2)}$, and if $(\gamma, \gamma_1)\in G^{(2)}$, then $\gamma^{-1}(\gamma\gamma_1) = \gamma_1$.

The maps r and d on G, defined by the formulae $r(\gamma) = \gamma\gamma^{-1}$ and $d(\gamma) = \gamma^{-1}\gamma$, are called the range and the source (domain) maps. They have a common image called the unit space of G, which is denoted $G^{(0)}$. For x and y in $G^{(0)}$, (r,d)-fibre is $G^x_y = G^x \cap G_y$. It is easily follows from the definition of the groupoid that G^x_x is a group, called the isotropy group at x. The relation x~y if and only if there is $\gamma\in G$ such that $r(\gamma)=x$ and $d(\gamma)$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits.

A topological groupoid G consists of a groupoid G and a topology compatible with the groupoid structure i.e. the inversion and product map are continuous map. (the topology on $G^{(2)}$ is coming from G×G endowed with the product topology).

The setting for the study of dynamical systems involves three ingredients: a space (phase space) X, time T, and a time evolution. In the classical case of discrete-time systems the time T is □ (reversible case) or □ (irreversible case).

In the autonomous case the time-evolution law is given by an action of T on phase space (the space of all possible states of the system). The action is defined by iterates of a generator $\sigma:X\to X$. The map σ defines time evolution in the following way: the state $x\in X$ at time t = 0 evolves into $\sigma(x)$ at t = 1, $\sigma(\sigma(x))$ at t = 2, etc. Consequently, $\sigma^n(x)$ is the state of the system at time t = n if x is the state of the system at time t = 0 is x. If σ is invertible, (X, σ) formalizes a reversible discrete-time process (dynamical system): the action of □ on

X is given by $n \bullet x = \sigma^n(x)$, where if $n < 0$ then $\sigma^n(x) = \sigma^{-1}(\sigma^{-1}\ldots(x))$ (σ^{-1} being the inverse of σ) and $\sigma^0(x) = x$. A groupoid associated to a discrete-time dynamical system (X, σ) was defined in [3] in the following way:.

$$G(X, \sigma) = \{(x, n-k, y) \in X \times \mathbb{Z} \times X : n, k \in \mathbb{Z}, \sigma^n(x) = \sigma^k(y)\}.$$

Under the operations:

$(x, n, y)(y, m, z) = (x, n+m, z)$ (product)

$(x, n, y)^{-1} = (y, -n, x)$ (inversion).

G is a indeed a groupoid. When $\sigma : X \to X$ is invertible, the map $(x, n, y) \to (x,n)$ is an isomorphism from $G(X,\sigma)$ onto transformation groupoid $\mathbb{Z} \times X$ defined by the $\mathbb{Z}$-action on X: $n \bullet x \to \sigma^n(x)$. Let us recall that $\mathbb{Z} \times X$ is a groupoid under the operations:

$(n, m \bullet x)(m, x) = (m+n\ x)$ (product)

$(n, x)^{-1} = (-n, n \bullet x)$ (inverse)

A category of such kind of discrete dynamical systems was studied in [1].

The mathematical formalization for a nonautonomous discrete-time process demands a space X and a sequence $(f_n)_n$ of maps $f_n: X \to X$. Then the nonautonomous difference equation

$$x_{n+1} = f_n(x_n)$$

generates a discrete-time process which is defined for all $x \in X$ and $n, n_0 \in \mathbb{Z}$ with $n \geq n_0$ by:

$\sigma(n_0, n_0, x) := x$,

$\sigma(n, n_0, x) := f_{n-1} \circ f_{n-2} \circ \ldots \circ f_{n_0}(x)$.

Then

1.1) $\sigma(n_0, n_0, x) := x$ for all $n_0 \in \mathbb{Z}$ and $x \in X$

1.2) $\sigma(n_2, n_0, x) := \sigma(n_2, n_1, \sigma(n_1, n_0, x))$ for all $n_0 \leq n_1 \leq n_2$ and $x \in X$

A process σ that satisfies semigroup property 1.2 can be reformulated as an autonomous irreversible dynamical system. The extended phase space is $\tilde{X} = \mathbb{Z} \times X$. The property 1.2 allows us to define an action of $\mathbb{Z}$ on $\tilde{X}$ by

$$n \bullet (n_0, x) := (n + n_0, \sigma(n + n_0, x))$$

for all $n \in \mathbb{Z}$ and $(n_0, x) \in \tilde{X}$.

Thus if a process σ that satisfies semigroup property 1.2, then some of its properties can also be studied in the groupoid framework.

A fractional order discrete time system has the form:

$$\Delta^{[\alpha]} x_{n+1} = g(x_n, n)$$

or equivalently,

$$x_{n+1} = \sum_{i=1}^{n+1} c_i x_{n+1-i} + g(x_n, n)$$

where

$$c_i = (-1)^{i+1} \binom{i}{\alpha} = (-1)^{i+1} \frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!}$$

Thus for this system the difference equation may be written as

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_0)$$

which generate a process that in general does not satisfies semigroup condition 1.2. In [2] can be found a fractional order system that do not satisfies semigroup property (in continuous case).

The purpose of this paper is to introduce a groupoid associated to a diference equation of the form

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_0)$$

More precisely the groupoid will be associated to an uniform space X and a function $f: X \times \mathbb{Z} \to X$ having the meaning that $f(x,n) = f_n(x_{n-1}, x_{n-2}, \ldots, x_0)$ with $x_0 = x$ (or $x_{n_0} = x$ for a fixed n_0).

2. A GROUPOID ASSOCIATED TO A GENERAL DISCRETE-TIME SYSTEM

Let us recall few aspects concerning the uniform spaces. A uniform space is a set X equipped with a nonempty family $\mathcal{A}$ of subsets of $X \times X$ (called uniform structure on X) satisfying the following conditions:

1. if $U \in \mathcal{A}$, then $\Delta \subset U$, where $\Delta = \{ (x, x) : x \in X \}$.
2. if $U \in \mathcal{A}$ and $U \subset V \subset X \times X$ then $V \in \mathcal{A}$

3. if $U\in\mathcal{A}$ and $V\in\mathcal{A}$, then $U\cap V\in\mathcal{A}$

4. if $U\in\mathcal{A}$, then there exists $V\in\mathcal{A}$ such that $VV\subset U$, where as usual

$$VV=\{(x,z): \text{there is } y \text{ such that } (x,y)\in V \text{ and } (y,z)\in V\}$$

5. if $U\in\mathcal{A}$, then $U^{-1} = \{ (y, x) : (x, y) \in U \} \in \mathcal{A}$.

If $\mathcal{A}$ is a uniform structure on X, then $\mathcal{A}$ induces a topology on X: $A\subset X$ is open if and only if for every $x\in A$ there exists $U\in\mathcal{A}$ such $\{y: (x,y) \in U\}\subset A$.

Proposition 2.1. Let $\mathcal{A}$ be a uniform structure on a set X and let f: X× □ →X be a function. If

$G(X,\mathcal{A},f)$ = {(x, k, y) ∈ X × □ × X :
for all $U\in\mathcal{A}$
there is $n_U\in$ □ such that
for all $n\geq n_U$, n+k∈ □ and
$(f(n+k,x),f(n,y))\in U\}$,

Then

1. $G(X,\mathcal{A},f)$ is a subgroupoid of X×□×X seen as a groupoid under the operations

(x, k, y)(y, m, z) = (x, k+m, z) (product)
$(x, k, y)^{-1}$ = (y, -k, x) (inversion).

2. If $G(X,\mathcal{A},f)$ is endowed with the induced topology from X×□×X, then $G(X,\mathcal{A},f)$ is a topological groupoid.

Proof. 1. Let (x, k, y)∈ $G(X,\mathcal{A},f)$ and let us prove that (y, -k, x)∈ $G(X,\mathcal{A},f)$. Let $U\in\mathcal{A}$. Then $U^{-1}\in\mathcal{A}$, hence there is $n_U\in$ □ such that for all $n\geq n_U$ n+k∈ □ and $(f(n+k,x),f(n,y))\in U^{-1}$. Thus for all $n\geq\max\{n_U+k,k\}$ we have n-k∈ □ and (f(n-k,y), f(n,k))∈U.

Consequently,

(y, -k, x)∈ $G(X,\mathcal{A},f)$.

Let (x, k, y), (y, m, z)∈ $G(X,\mathcal{A},f)$ and let us prove that (x, k+m, z)∈G. Let $U\in\mathcal{A}$. Then $V\in\mathcal{A}$ such that $VV\subset U$. Since (x,k,y)∈ $G(X,\mathcal{A},f)$ there is $n_V\in$ □ such that for all $n\geq n_V$, n+k∈ □ and (f(n+k,x),f(n,y))∈V. Since (y, m, z)∈ $G(X,\mathcal{A},f)$, there is $n'_V\in$ □ such that for all $n\geq n'_V$, n+m∈ □ and

(f(n+m,y),f(n,z))∈ $G(X,\mathcal{A},f)$.

Thus for all $n\geq\max\{n_V-m, n'_V\}$ we have n+k+m∈ □ , (f(n+k+m,x), f(n+m,y))∈V and (f(n+m,y), f(n,z))∈V. Thus

(f(n+k+m,x),f(n+m,y))(f(n+m,y), f(n,z)) $\in VV\subset U$

for all $n\geq\max\{n_V-m, n'_V\}$. Consequently, (x, k+m, z)∈ $G(X,\mathcal{A},f)$.

2 . Since X×□×X is a topological groupoid, and $G(X,\mathcal{A},f)$ a subgroupoid, it follows that $G(X,\mathcal{A},f)$ is a topological groupoid,

Definition 2.2. An element x∈X is said to be an equilibrium point of the system defined by f: X×□ →X if there is $n_0\in$ □ such that f(x,n)=x for all n∈ □ , $n\geq n_0$.

Remark 2.3. If x is an equilibrium point of the system defined by f: X×□ →X and G is the groupoid defined in Proposition 2.1 then G_x^x = □ .

Lemma 2.4. Let $\mathcal{A}$ be a uniform structure on X, let f: X×□ →X be a function and let x_e be an equilibrium point of the system associated with f. Then x and x_e are equivalent units of $G(X,\mathcal{A},f)$ if and only if $\lim_{n\to\infty} f(n,x) = x_e$.

Proof. x and x_e are equivalent units of $G(X,\mathcal{A},f)$ if and only if there is k∈ □ such that $(x_e,k,x)\in G(X,\mathcal{A},f)$ if and only if for every $U\in\mathcal{A}$ there is $n_U\in$ □ such that for all $n\geq n_U$, n+k∈ □ and $(f(n+k,x_e),f(n,x))\in U$. Since x_e is an equilibrium point, $f(n+k,x_e)=x_e$ for all n and k such that $n+k\geq n_0$. It follows that $(x_e,f(n,x))\in U$ for all $n\geq\max\{n_U,n_0-k\}=n'_U$. Thus $(x_e,k,x)\in G(X,\mathcal{A},f)$ if and only if for every $U\in\mathcal{A}$ there is $n_U\in$ □ such that for all $n\geq n'_U$, $(x_e,f(n,x))\in U$ or equivalently, $\lim_{n\to\infty} f(n,x) = x_e$.

Corollary 2.5. Let $\mathcal{A}$ be a uniform structure on X such that

$$\bigcap_{U\in\mathcal{A}} U = \{ (x, x) : x \in X \}.$$

If f: $X\times\mathbb{N} \to X$ is a function, then each orbit of the groupoid $G(X,\mathcal{A},f)$ contains at most an equilibrium point of the system associated with f.

Proof. Let be x and y two equivalent units of $G(X,\mathcal{A},f)$. Let us assume that x and y are equilibrium points. Then $x=\lim_{n\to\infty} f(n,y)=\lim_{n\to\infty} y = y$ (since in this case the topology on X induced by the uniform structure $\mathcal{A}$ is Hausdorff).

Definition 2.6. Let us consider a system defined by a function f: $X\times\mathbb{N} \to X$ and let us assume that X is endowed with a uniform structure $\mathcal{A}$. An equilibrium point x_e of the system is said to be stable if there is $n_0\in\mathbb{N}$ such that for every $U\in\mathcal{A}$ there is $V_U\in\mathcal{A}$ with the property that if $(x_e,x)\in V_U$, then $(x_e, f(n,x))\in U$ for all $n\in\mathbb{N}$, $n\geq n_0$.

Remark 2.7. The x_e equilibrium point is stable if there is a neighborhood $E\subset X$ of x_e such that $f|_{E\times\{n:n\geq n_0\}}$ is uniformly continuous.

Definition 2.8. Let us consider a system defined by a function f: $X\times\mathbb{N} \to X$ and let us assume that X is endowed with a uniform structure $\mathcal{A}$.

- An equilibrium point x_e is said to be attractive if there is $U\in\mathcal{A}$ such that if $(x_e,x)\in U$, then $\lim_{n\to\infty} f(n,x) = x_e$.
- An equilibrium point x_e is asymptotically stable if it is stable and attracting.

Proposition 2.9. Let $\mathcal{A}$ be a uniform structure on X, let f: $X\times\mathbb{N} \to X$ be a function and let x_e be an equilibrium point. Then x_e is attractive if and only if x_e is in interior of its orbit with respect to the structure of the groupoid $G(X,\mathcal{A},f)$. This means that if

$$[x_e]=\{x\in X: \text{there is } k\in\mathbb{Z} \text{ such that } (x_e,k,x) \in G(X,\mathcal{A},f)\},$$

then x_e is attractive if and only if x_e belongs to the interior of $[x_e]\subset X$.

Proof. Let us assume that x_e belongs to the interior of $[x_e]\subset X$. Then there is $U\in\mathcal{A}$ such that $\{x: (x_e, x)\in U\}\subset[x_e]$. Let $x\in X$ such that $(x_e, x)\in U$. Then $x\in[x_e]$ and according Lemma 2.4, $\lim_{n\to\infty} f(n,x) = x_e$. Thus x_e is attractive. Conversely, assume that x_e is attractive. Then there is $U\in\mathcal{A}$ such that if $(x_e,x)\in U$, then $\lim_{n\to\infty} f(n,x) = x_e$. By Lemma 2.4, if $\lim_{n\to\infty} f(n,x) = x_e$, then $x\in[x_e]$. Hence

$$\{x: (x_e, x)\in U\}\subset[x_e].$$

Therefore x_e belongs to the interior of $[x_e]\subset X$.

Corollary. Let $\mathcal{A}$ be a uniform structure on a X, let f: $X\times\mathbb{N} \to X$ be a function and let x_e be a stable equilibrium point. Then x_e is asymptotically stable if and only if x_e is in interior of its orbit with respect to the structure of the groupoid $G(X,\mathcal{A},f)$ (defined in Proposition 2.1)

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