BISIMILAR SUBGROUPOIDS OF TRIVIAL GROUPOIDS WITH GROUP Z

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ABSTRACT: The purpose of this short note is to prove that if the category of all subgroupoids of trivial groupoids with group $\mathbb{Z}$ is seen as a model category $\mathcal{M}$ in the sense [M. Nielsen A. Joyal and G. Winskel, 1996] and [E. Haghverdi, P. Tabuada, G.J. Pappas, 2002], then we can highlight a path category $\mathcal{P}$ (a subcategory of $\mathcal{M}$ serving as an abstract notion of time) such that the $\mathcal{P}$-open morphisms are exactly the groupoid homomorphisms.

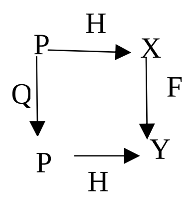
KEY WORDS: trivial groupoid; groupoid homomorphism; bisimulation.

1. INTRODUCTION AND PRELIMINARIES

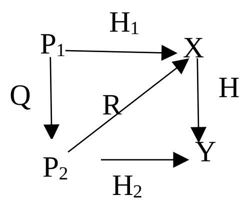
The notion of bisimilarity is connected to complexity reduction of dynamical systems and to the problem of equivalence of systems. We use in this paper the abstract setting introduced in [7] where it is considered a category of models where the objects are the systems being studied and the morphisms are simulations. For the necessary concepts concerning Category Theory see for instance [6]. More precisely, the framework used in [7] and [5] consists of two components:

- Model Category: The model category $\mathcal{M}$ with objects the systems and morphisms simulations in the sense that a morphism $F : X \to Y$ in $\mathcal{M}$ should be thought of as a simulation of system X in system Y.
- Path Category: The path category $\mathcal{P}$ (serve as an abstract notion of time) is a subcategory of $\mathcal{M}$ of path objects and with morphisms expressing how they can be extended.

The abstract notion of bisimulation [7] requires the notion of $\mathcal{P}$-open morphism. Let us recall that a morphism $F : X \to Y$ in $\mathcal{M}$ is said to be $\mathcal{P}$-open if for every morphisms $H_1:P_1\to X$ and $H_2:P_2\to Y$ in $\mathcal{M}$ and every morphism $Q:P_1\to P_2$ in $\mathcal{P}$ such that the following diagram commutes (i.e. $FH_1=H_2Q$):



there is a morphism $R:P_2\to X$ in $\mathcal{M}$ such that in the below diagram both triangles commute



(i.e $RQ=H_1$ and $HR= H_2$)

2. THE CATEGORY OF SUBGROUPOIDS OF TRIVIAL GROUPOIDS WITH GROUP $\mathbb{Z}$ AS MODEL CATEGORY

In [3] we have explained the conection between a discrete dynamical system and a subgroupoid of a trivial groupoid with group $\mathbb{Z}$ via the groupoid studied in [4]. In this paper we use the notation in [1], [2] and [3]. As in [1] (Section 2) and [2] any subgroupoid G of the trivial groupoid is represented by $X\times\mathbb{Z}\times X$ can be represented as $G = \bigcup_{f(u)=f(v)} G_v^u$ with

$$G_v^u=\{(u, nk(f(u))+k(u)-k(v),v): n\in\mathbb{Z}\},$$

where $f: X\to X$ and $k:X\to\mathbb{Z}$ are two functions with certain properties. This groupoid is denoted $G=G(X,f,k)$. We also denote

$$\bar{X} =\{[u],u\in G^{(0)}=X\}$$

According [3] (Corollary 4) a groupoid homorphism $H: G(X,f_X,k_X) \to G(Y,f_Y,k_Y)$ is determined and determines three functions:

- $h:X\to Y$ with the property that $$h([u])\subset[h(u)]$$ for all $u\in X$
- $\mu:X\to\mathbb{Z}$
- $\eta:\bar{X}\to\mathbb{Z}$

The functions h, η are uniquely determined by $H: G(X,f_X,k_X) \to G(Y,f_Y,k_Y)$ and μ is determined modulo a function $\mu_1: \bar{X}\to\mathbb{Z}$. The homomorphism H defined by the three functions by

$$H(u,nk_X(f_X(u))+k_X(u)-k_X(v),v) = $$
$$=(h(u), (n\eta([u])+\mu(u)-\mu(v))k_Y(f_Y(h(u))) + k_Y(h(u))- k_Y(h(v)),h(v))$$

is denoted $H=H(h,\eta,\mu)$.

Proposition 1. Let $X_1\times\mathbb{Z}\times X_1$ and $X_2\times\mathbb{Z}\times X_2$ and $X_3\times\mathbb{Z}\times X_3$ be three trivial groupoids with group $\mathbb{Z}$ and $G_i= G(X_i,f_i,k_i)$ be a subgroupoid of $X_i\times\mathbb{Z}\times X_i$ for $i\in\{1,2,3\}$. If

$H_1=H(h_1,\eta_1,\mu_1): G(X_1,f_1,k_1) \to G(X_2,f_2,k_2)$

$H_2=H(h_2,\eta_2,\mu_2): G(X_2,f_2,k_2) \to G(X_3,f_3,k_3)$

are two a groupoid homomorphisms, then

$$H_2\circ H_1=H(h_3,\eta_3,\mu_3)$$

where

$h_3 = h_2\circ h_1$

$\eta_3([u]) = \eta_1([u])\, \eta_2([h_1(u)])$

$\mu_3(u) = \mu_1(u)\, \eta_2([h_1(u)]) +\mu_2(h_1(u))$

for all $u \in X_1$.

Proof. If

$(u,nk_1(f_1(u))+k_1(u)-k_1(v),v)\in G(X_1,f_1,k_1)$ then

$H_2\circ H_1(u,nk_1(f_1(u))+k_1(u)-k_1(v),v)=$
$H_2(h_1(u),(n\eta_1([u])+\mu_1(u)-\mu_1(v))$
$k_2(f_2(h_1(u)))+k_2(h_1(u))- k_2(h_1(v)),h_1(v)))=$
$(h_2(h_1(u)), ((n\eta_1 ([u]+\mu_1(u)-\mu_1(v))$
$\eta_2([h_1(u)]) +\mu_2(h_1(u))-\mu_2(h_1(v)))$
$k_3(f_3(h_2(h_1(u))))+k_3(h_2(h_1(u)))-$
$k_3(h_2(h_1(v))),h_2(h_1(v)))) =$
$(h_2(h_1(u)), (n\eta_1 ([u])\, \eta_2([h_1(u)])+\mu_1(u)$
$\eta_2([h_1(u)] +\mu_2(h_1(u))-\mu_1(v)\, \eta_2([h_1(u)]) -$
$\mu_2(h_1(v)))\, k_3(f_3(h_2(h_1(u))))+k_3(h_2(h_1(u)))-$
$k_3(h_2(h_1(v))), h_2(h_1(v))))$.

Thus we obtain

$h_3= h_2\circ h_1$

$\eta_3([u])=\eta_1([u])\, \eta_2([h_1(u)])$

$\mu_3(u)= \mu_1(u)\, \eta_2([h_1(u)]) +\mu_2(h_1(u))$.

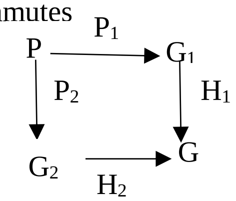
Notation 2. The category of subgroupoids of trivial groupoids with group $\mathbb{Z}$ as objects and groupoid homomorphisms as morphism is denoted $\mathcal{M}$ and serves as the model category.

Proposition 3. Let $G=G(X,f,k)$, $G_1=G(X_1,f_1,k_1)$ and $G_2=G(X_2,f_2,k_2)$ be three objects in $\mathcal{M}$ and

$H_1=H(h_1,\eta_1,\mu_1): G(X_1,f_1,k_1)\to G(X,f,k)$

$H_2=H(h_2,\eta_2,\mu_2): G(X_2,f_2,k_2)\to G(X,f,k)$

be morphisms in $\mathcal{M}$. Then there is an object P in $\mathcal{M}$ and two morphism $P_1:P\to G(X_1,f_1,k_1)$ and $P_2:P\to G(X_2,f_2,k_2)$ such that the following diagram commutes



Proof. Let us define $\tilde{X}$ is the set all all $(u_1,u_2)\in X_1\times X_2$ satisfying the following properties:

1. $h_1(u_1)=h_2(u_2)$
2. there are $\xi_1(u_1,u_2),\ \xi_2(u_1,u_2)\in\mathbb{Z}$ such that

$$\xi_1(u_1,u_2)\,\eta_1([u_1]) +\mu_1(u_1) =$$
$$=\xi_2(u_1,u_2)\,\eta_2([u_2]) +\mu_2(u_2)$$

$$P=\{((u_1,u_2),n,(v_1,v_2))\in \tilde{X}\times\mathbb{Z}\times\tilde{X} : f_1(u_1)=f_1(v_1),\ f_2(u_2)=f_2(v_2)\}$$

Then P is a subgroupoid of $\tilde{X}\times\mathbb{Z}\times\tilde{X}$ (the trivial groupoid on $\tilde{X}$ with group $\mathbb{Z}$). Consequently, there are $\tilde{f}: \tilde{X}\to\tilde{X}$ and $\tilde{k}: \tilde{X}\to\mathbb{Z}$, such that $P= G(\tilde{X},\tilde{f},\tilde{k})$. Let us define the $P_1=H(pr_1, \eta_1^p , \mu_1^p)$, $P_2=H(pr_2, \eta_2^p, \mu_2^p)$, where

$\eta_1^p([(u_1,u_2)]=\eta_2([u_2])$,

$\eta_2^p([(u_1,u_2)]=\eta_1([u_1])$,

$\mu_1^p(u_1,u_2)= \xi_1(u_1,u_2)$,

$\mu_2^p(u_1,u_2)= \xi_2(u_1,u_2)$.

Then $H_1 \circ P_1=H_2 \circ P_2$.

Definition 4. The objects of the path category $\mathcal{P}$ (serving as an abstract notion of time) are groupoids of the form

$$\coprod_{i\in I}\{(n,n-m,m):(n,m)\in S_i\times S_i\}$$

where $S_i\subset\mathbb{N}$ for all $i\in I$. The morphisms are groupoid homomorphism whose restrictions to units spaces are bijective.

Remark 5. Let us consider

$$G=\coprod_{i\in I}\{(n,n-m,m):(n,m)\in S_i\times S_i\}$$

an object in the path category. Then it is easy to see that $G=G(\coprod_i S_i, f, k)\subset\mathbb{N}\times\mathbb{Z}\times\mathbb{N}$ where for each $n\in\coprod_i S_i$ $f(n)$ is the first (minimum) element of S_i with $n\in S_i$ and $k(n)=n-f(n)$ for all $n\neq f(n)$. For this groupoid $k(f(n))=0$ for all n. That is why for any groupoid homomorphism $H=H(h,\eta,\mu)$ (morphism in $\mathcal{M}$) having the domain in $\mathcal{P}$, η can be taken to be 0.

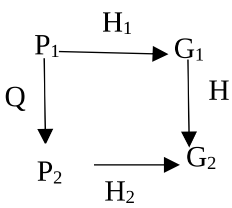
Proposition 6. Let $G_1=G(X_1,f_1,k_1)$ and $G_2=G(X_2,f_2,k_2)$ be two objects in $\mathcal{M}$. Let $H(h,\eta,\mu)$: $G_1\to G_2$ be a morphism in $\mathcal{M}$ (a groupoid homomorphism). Then $H(h,\eta,\mu)$ is $\mathcal{P}$-open.

Proof. Let us consider P_1, P_2 two objects in the path category and $Q=H(\tau,0,\xi)$: $P_1\to P_2$ a homomorphism in the same category. Let us also consider three groupoids G, G_1 and G_2 in the category of models and two groupoid homomorphisms

$H_1=H(h_1,0,\mu_1):P_1\to G_1$

$H_2=H(h_2,0,\mu_2):P_2\to G_2$

such that the following diagram commutes:



This means $H\circ H_1=H_2\circ Q$ or equivalently,

$h\circ h_1= h_2\circ\tau$

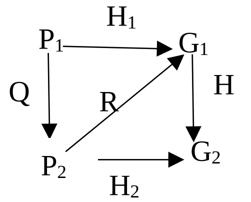
$\mu_1(u)\ \eta([h_1(u)]) +\mu(h_1(u)) = \mu_2(\tau(u))$

Since τ is bijective, there is τ': $\mathbb{Z}\to\mathbb{Z}$ such that $\tau'\circ\tau=\tau\circ\tau'=1_{\mathbb{Z}}$.

Let $R=H(h_3,0,\mu_3)$ be defined by

$h_3=h_1\circ\tau'$ and $\mu_3=\mu_1\circ\tau'$

We prove that in the following diagram both triangles commute:



We have

$h_3\circ\tau= h_1$

$\mu_3(\tau(u)) = \mu_1(u)$.

Consequently, $R\circ Q=H_1$. Also

$h\circ h_3 = h\circ h_1\circ\tau'=h_2\circ\tau\circ\tau'=h_2$

and since

$\mu_1(u)\ \eta([h_1(u)]) +\mu(h_1(u)) = \mu_2(\tau(u))$

it follows that

$\mu_1(\tau'(u'))\ \eta([h_1(\tau'(u'))]) +\mu(h_1(\tau'(u'))) =$

$= \mu_2(\tau(\tau'(u')))$

Hence

$\mu_3(u')\ \eta([h_3(u')]) +\mu(h_3(u')) = \mu_2(u')$.

and therefore $H\circ R=H_2$. Thus H is $\mathcal{P}$-open.

Definition 7. Let us consider $\mathcal{C}$ a subcategory of $\mathcal{M}$ with the properties

- $Ob(\mathcal{C})=Ob(\mathcal{M})$
- if $P:G_1\to G_2$ is a surjective groupoid homorphism) and $Q:G_2\to G_3$ a morphism in **C** then QP is a morphism in **C**. Two groupoids G_1 and G_2 are called $\mathcal{C}$-bisimilar if there is another groupoid G and two morphisms in **C**:

$H_1=H(h_1,\eta_1,\mu_1):G\to G_1$

$H_2=H(h_2,\eta_2,\mu_2):G\to G_2$.

Proposition 8. The relation of $\mathcal{C}$-bisimilarity introduced in Definition 7 is an equivalence relation.

Proof. Since the identity is a morphism in $\mathcal{C}$, it follows that the $\mathcal{C}$-bisimilarity is reflexive. It is obviously, symmetric. It remains to prove the transitivity. Let us assume that G_1 and G_2 are $\mathcal{C}$-bisimilar and also

G_2 and G_3 are $\mathcal{C}$-bisimilar. There are two groupoids G_4 and G_5 and four morphisms in $\mathcal{C}$: $H_1:G_4\to G_1$, $H_2:G_4\to G_2$, $H_3:G_5\to G_2$ and $H_4:G_5\to G_3$.
Similarly, as in the proof of Proposition 3 let

$$\tilde{X} = X_4\times X_5$$

and

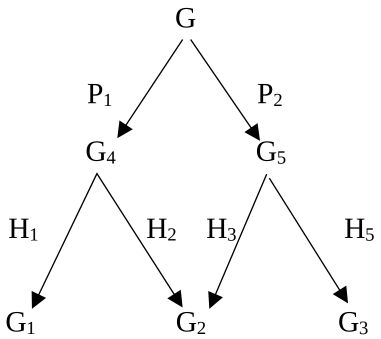
$$G=\{((u_1,u_2),n,(v_1,v_2))\in \tilde{X}\times\mathbb{Z}\times\tilde{X} : f_4(u_1)=f_4(v_1),\ f_5(u_2)=f_5(v_2)\}$$

Obviously, G is a subgroupoid of $\tilde{X}\times\mathbb{Z}\times\tilde{X}$ (seen as the trivial groupoid on $\tilde{X}$ with group $\mathbb{Z}$). Consequently, there are $\tilde{f}:\tilde{X}\to\tilde{X}$ and $\tilde{k}:\tilde{X}\to\mathbb{Z}$, such that $G= G(\tilde{X},\tilde{f},\tilde{k})$. Let us consider the following groupoid homomorphisms

$$P_1=H(pr_1,1\ ,0):G\to G_4$$

$$P_2=H(pr_2,\ 1,0):G\to G_5$$

which are surjective homomorphisms.



Then $H_1\circ P_1:G\to G_1$ and $H_2\circ P_2:G\to G_2$ are groupoid homomorphisms and morphisms in $\mathcal{C}$.

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