PERTURBED DISCRETE-TIME RICCATI EQUATIONS OF CONTROL

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ABSTRACT: In this paper we shall prove that a class of perturbed discrete-time Riccati equations (DTREs) have solutions which are independent of perturbations. These Riccati equations arise in optimal control problems associated with systems with memory.

KEYWORDS:: discrete-time Riccati equation of control, optimal control, stochastic equations

1. Introduction.

Riccati equations play an important role in control theory. For more details we refer the reader to [1-4], [8] and the references therein. In this paper we investigate the solution properties of a family of perturbed, backward DTRE**s** associated with a class of linear systems with memory which are closely related to linear discrete-time fractional order systems [5], [6].

2. Preliminaries

Let $\{\xi_k\}_{k\in\mathbf{N}}$ be a sequence of real-valued, mutually independent random variables on the probability space $(\Omega,\mathcal{G},P)$ that satisfies the condition $E[\xi_k]=0, E[\xi_k^2]=1, k\in\mathbf{N}$. (Here $E[\xi]$ is the mean (expectation) of ξ_k .) The $\sigma-$ algebra generated by $\{\xi_i, 0\le i\le n-1\}, n\in\mathbf{N}^*$. Assume that $N\in N$ is fixed and A_i , i = 1,..,N, $B\in R^{d\times d}$, D , F $\in R^{d\times m}, m\in N$. Consider the following system

$$x_{k+1}=\sum_{j=0}^{k}A_j x_{k-j}+\xi_k Bx_k+Du_k+\xi_k Fu_k, \quad (2.1)$$
$$x_0=x\in R^d,$$

where the control variable $u=\{u_k\}_{k\in N}$ belongs to $\mathcal{U}^a\in\{u=\{u_k\}_{k\in N}$, u_k are $\mathcal{G}_k$ -measurable, R^m -valued random variables satisfying $E\left[\|u_k\|^2\right]<\infty$ for all $k\in N\}$. Here $\mathcal{G}_n$ is the $\sigma-$ algebra generated by $\{\xi_i, 0\le i\le n-1\}, n\in\mathbf{N}^*$. For any $x_0\in R^d$ and $N\in N$, fixed and $C\in R^{p\times d}, S\in R^{d\times d}, S\ge 0, K\in R^{m\times m}, K>0$ we consider the optimal control problem $\mathcal{O}$ which consist in minimizing the cost functional

$$I_{x_0,N}(u)=E<Sx_N,x_N>+\sum_{n=0}^{N-1}E\left[\left(\|Cx_n\|^2+<Ku_n,u_n>\right)\right] \quad (2.2)$$

subject to (1.1), over the class $\mathcal{U}^a_{0,N-1}\subseteq\mathcal{U}^a_1$ of segments u = { u_0, u_1, u_2,..,u_{N-1}, 0, 0,..} of admissible controls.

3. An equivalent linear form of the system with memory

Let $\mathcal{A},\mathcal{B}:(R^d)^N\to(R^d)^N$ be the linear operators defined by the matrices

$$\mathbf{A} = \begin{pmatrix} A_0 & c_1 I_{R^d} & \dots & c_{N-1} I_{R^d} \\ I_{R^d} & 0 & . & 0 \\ . & I_{R^d} & . & . \\ . & . & I_{R^d} & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} B & 0 & . & . & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}.$$

For all $(v_0, v_1, ..., v_{N-1}) \in (R^m)^N$ let us define the following linear operators.

$$\mathbf{D}_k(v_0, v_1, ..., v_{N-1}) = (Dv_k, 0, ..., 0) \in (R^d)^N$$
$$\mathbf{F}_k(v_0, v_1, ..., v_{N-1}) = (Fv_k, 0, ..., 0) \in (R^d)^N$$
$$\mathbf{K}_k(v_0, v_1, ..., v_{N-1}) = (0, ..., Kv_k, 0, ..., 0) \in (R^m)^N$$

Similarly, for all $(v_0, v_1, ..., v_{N-1}) \in (R^d)^N$ we introduce the linear operators

$$\mathbf{C}(v_0, v_1, ..., v_{N-1}) = (Cv_0, 0, ..., 0)$$
$$\mathbf{S}(v_0, ..., v_{N-1}) = (Sv_0, 0, ..., 0).$$

Thus, $\mathbf{K}_k, \mathbf{S} \geq 0$. Let $x_0, x_1, ..., x_k, ...$ be a solution of (1.1). For any $k < N, X_k^T = (x_k, x_{k-1}, ..., x_0, 0, .., \underset{N}{0}) \in (R^d)^N$ is a solution of the linear system

$$X_{k+1} = \mathbf{A}X_k + \xi_k \mathbf{B}X_k + \mathbf{D}_k U_k + \xi_k \mathbf{F}_k U_k, \quad X_0 = (x_0, 0, ..., \underset{N}{0}), \tag{3.1}$$

where the control $U = \{U_k\}_{k \in N} \subset (R^m)^N$ belongs to the set U^a of admissible controls sequences $\{U_k\}_{k \in N}$ having the property that U_k are $(R^m)^N$ -valued, $\mathbf{G}_k$ -measurable random variables satisfying $E\left[\|U_k\|^2\right] < \infty$ for all k ⍰ $\mathbf{C}$. So, (2.1) is an equivalent form of (3.1). Since

$$\mathbf{C}X_k = \mathbf{C}(x_k, x_{k-1}, ..., x_0, 0, ..0) = (Cx_k, 0, ..., 0),$$
$$\mathbf{K}_k U_k = \mathbf{K}_k(\bar{u}_0, \bar{u}_1, ..., u_k .., \bar{u}_{N-1}) = (0, 0, ..., \underset{k+1}{Ku_k}, ..., 0).$$

the cost functional (2.2) can be equivalently rewritten as

$$I_{x_0,N}(U) = E[\sum_{k=0}^{N-1} \langle \mathbf{C}^* \mathbf{C} X_k, X_k \rangle + \langle \mathbf{S} X_N, X_N \rangle + \langle \mathbf{K}_k U_k, U_k \rangle]. \tag{3.2}$$

Substituting X_N given by (3.1) in (3.2), we get

$$I_{x_0,N}(U) = \sum_{n=0}^{N-2} E[\langle \mathbf{C}^* \mathbf{C} X_k, X_k \rangle + \langle \mathbf{K}_k U_k, U_k \rangle]$$
$$+ E[\langle (\mathbf{C}^*\mathbf{C} + \mathbf{A}^*\mathbf{S}\mathbf{A} + \mathbf{B}^*\mathbf{S}\mathbf{B}) X_{N-1}, X_{N-1} \rangle$$
$$+ 2\langle (\mathbf{D}_{N-1}^* \mathbf{S}\mathbf{A} + \mathbf{F}_{N-1}^* \mathbf{S}\mathbf{B}) X_{N-1}, U_{N-1} \rangle +$$
$$\langle (\mathbf{K}_{N-1} + \mathbf{D}_{N-1}^* \mathbf{S} \mathbf{D}_{N-1} + \mathbf{F}_{N-1}^* \mathbf{S} \mathbf{F}_{N-1}) U_{N-1}, U_{N-1} \rangle].$$

Now let $\mathsf{U}_{0,N-1}^a$ be the class of all finite segments $U_0, ..., U_{N-1}$ of sequences $U \in \mathsf{U}^a$. It is not difficult to see that the optimal control problem $\mathcal{O}$ is equivalent with the minimizing optimal control problem $\mathbf{O}_1$ defined by system (3.1), $I_{x_0,N}(U)$ and $\mathsf{U}_{0,N-1}^a$. Indeed, for any $u \in \mathbf{U}_{0,N-1}^a$, the segment

$$U = \{U_k = (0, .., \underset{k+1}{u_k}, ...0), k = 0, .., N-1\}$$

belongs to $\mathsf{U}_{0,N-1}^a$ and $I_{x_0,N}(U) = I_{x_0,N}(u)$. Conversely, given $U \in \mathsf{U}_{0,N-1}^a$, we define $u = \{u_k = U_{kk}, k = 0, .., N-1\}$. Thus, $u \in \mathbf{U}_{0,N-1}^a$ and $I_{x_0,N}(U) = I_{x_0,N}(u)$. Now it is clear that $\bar{U}$ is optimal for $I_{x_0,N}$⍰U⍰ if and only if $\bar{u} = \{\bar{u}_k = \bar{U}_{kk}, k = 0, .., N-1\}$ is optimal for $I_{x_0,N}(u)$ and $I_{x_0,N}(\bar{U}) = I_{x_0,N}(\bar{u})$.

The problem $\mathbf{O}_1$ is a linear quadratic optimal control problem for stochastic systems. However, the coefficient $\mathbf{K}_k$ of the cost functional $I_{x_0,N}(U)$ does not satisfy the condition $\mathbf{K}_k > 0$, $k = 0,.., N-1$ and we cannot solve $\mathbf{O}_1$ by a direct application of the known results from the optimal control theory of stochastic discrete-time systems (see [2], [3]).

Therefore, we replace the optimal cost $I_{x_0,N}(U)$ from $\mathbf{O}_1$ with the optimal cost

$$I_{x_0,N,\varepsilon}(U) = \sum_{k=0}^{N-2} E\left[\left\langle \mathbf{C}^*\mathbf{C}X_k, X_k\right\rangle\right] + E\left[\left\langle (\mathbf{K}_k + \varepsilon\mathbf{I}_k)U_k, U_k\right\rangle\right] + E[\left\langle \left(\mathbf{C}^*\mathbf{C} + \mathbf{A}^*\mathbf{S}\mathbf{A} + \mathbf{B}^*\mathbf{S}\mathbf{B}\right)X_{N-1}, X_{N-1}\right\rangle + 2\left\langle \left(\mathbf{D}_{N-1}^*\mathbf{S}\mathbf{A} + \mathbf{F}_{N-1}^*\mathbf{S}\mathbf{B}\right)X_{N-1}, U_{N-1}\right\rangle + \left\langle K_{N-1}^{\varepsilon}U_{N-1}, U_{N-1}\right\rangle]$$

where $\varepsilon > 0$ is fixed,

$$K_{N-1}^{\varepsilon} = \mathbf{K}_{N-1} + \varepsilon\mathbf{I}_{N-1} + \mathbf{D}_{N-1}^*\mathbf{S}\mathbf{D}_{N-1} + \mathbf{F}_{N-1}^*\mathbf{S}\mathbf{F}_{N-1}$$

and

$$\mathbf{I}_k(v_0, v_1, ..., v_{N-1}) = (v_0, v_1, ..., \underset{k+1}{0}, ..., v_{N-1}).$$

We have obtained a new optimal control problem $\mathbf{O}_\varepsilon$. The hypothesis $K > 0$, implies that $\mathbf{K}_k + \varepsilon\mathbf{I}_k > 0$, for all $k = 0,.., N-1$ and we can apply classical results to solve $\mathbf{O}_\varepsilon$.

4. Perturbed Riccati equation of control

Let us associate with $\mathbf{O}_\varepsilon$ the backward discrete-time Riccati equation

$$R_n^{\varepsilon} = \mathbf{A}^*R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{B}R_{n+1}^{\varepsilon}\mathbf{B} + \mathbf{C}^*\mathbf{C} - \left(\mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{B}\right)^* \cdot (\mathbf{K}_n + \varepsilon\mathbf{I}_n + \mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{F}_n)^{-1} \left(\mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{B}\right), n < N-1 \quad (4.1)$$

$$R_{N-1}^{\varepsilon} = \mathbf{C}^*\mathbf{C} + \mathbf{A}^*\mathbf{S}\mathbf{A} + \mathbf{B}^*\mathbf{S}\mathbf{B} - \left(\mathbf{D}_{N-1}^*\mathbf{S}\mathbf{A} + \mathbf{F}_{N-1}^*\mathbf{S}\mathbf{B}\right)^* \cdot \left(K_{N-1}^{\varepsilon}\right)^{-1} \left(\mathbf{D}_{N-1}^*\mathbf{S}\mathbf{A} + \mathbf{F}_{N-1}^*\mathbf{S}\mathbf{B}\right).$$

An easy computation and formula (4.8) from [8] shows that

$$R_n^{\varepsilon} = \left(\mathbf{A} + \mathbf{D}_nW_n\right)^* R_{n+1}^{\varepsilon}\left(\mathbf{A} + \mathbf{D}_nW_n\right) + \left(B + \mathbf{F}_nW_n\right)^* R_{n+1}^{\varepsilon}\left(B + \mathbf{F}_nW_n\right) + \mathbf{C}^*\mathbf{C} + W_n^*\left(\mathbf{K}_n + \varepsilon I_n\right)W_n, n = 0,.., N-2.$$

$$R_{N-1}^{\varepsilon} = \mathbf{C}^*\mathbf{C} + \left(\mathbf{A} + \mathbf{D}_{N-1}W_{N-1}\right)^*\mathbf{S} \bullet \left(\mathbf{A} + \mathbf{D}_{N-1}W_{N-1}\right) + \left(\mathbf{B} + \mathbf{F}_{N-1}W_{N-1}\right)^*\mathbf{S}\left(\mathbf{B} + \mathbf{F}_{N-1}W_{N-1}\right) + W_{N-1}^*\left(\mathbf{K}_{N-1} + \varepsilon\mathbf{I}_{N-1}\right)W_{N-1},$$

for

$$W_{N-1} = -\left(K_{N-1}^{\varepsilon}\right)^{-1}\left(\mathbf{D}_{N-1}^*\mathbf{S}\mathbf{A} + \mathbf{F}_{N-1}^*\mathbf{S}\mathbf{B}\right), \quad (4.2)$$

and

$$W_n = -(\mathbf{K}_n + \mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{F}_n + \varepsilon I_n)^{-1} \cdot \left(\mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{B}\right), n = 0,.., N-2. \quad (4.3)$$

Now it is clear that $R_n^{\varepsilon} \geq 0, n = 0,.., N-1$.

Lemma 1 The cost functional $I_{x_0,N,\varepsilon}(U$ can be equivalently rewritten as

$$I_{x_0,N,\varepsilon}(U) = E\left[\left\langle R_0^{\varepsilon}X_0, X_0\right\rangle\right] + \left\langle K_{N-1}^{\varepsilon}\left(W_{N-1}X_{N-1} - U_{N-1}\right), \left(W_{N-1}X_{N-1} - U_{N-1}\right)\right\rangle + \sum_{k=0}^{N-2} E\left[\left\|\left(\mathbf{K}_n + \mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{F}_n + \varepsilon\mathbf{I}_n\right)^{1/2}\left(W_nX_n - U_n\right)\right\|^2\right],$$

where R_n^{ε} is the unique solution of (4.1).

Proof. Let X_{n+1} be defined by (3.1). We have

$$E\left[\left\langle R_{n+1}^{\varepsilon}X_{n+1}, X_{n+1}\right\rangle\right] = E[\left\langle \left(\mathbf{A}^*R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{B}R_{n+1}^{\varepsilon}\mathbf{B}\right)X_n, X_n\right\rangle + 2\left\langle \left(\mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{A} + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{B}\right)X_n, U_n\right\rangle + \left\langle \left(\mathbf{D}_n^*R_{n+1}^{\varepsilon}\mathbf{D}_n + \mathbf{F}_n^*R_{n+1}^{\varepsilon}\mathbf{F}_n\right)U_n, U_n\right\rangle]$$

and, taking into account (4.1) and (4.3), we obtain

$$E\left[\left\langle R^{\varepsilon}_{n+1}X_{n+1},X_{n+1}\right\rangle\right]=E\left[\left\langle R^{\varepsilon}_{n}X_{n},X_{n}\right\rangle-\left\langle \mathbf{C}^{*}\mathbf{C}X_{n},X_{n}\right\rangle\right]-$$
$$E\left[\left\langle(\mathbf{K}_n+\varepsilon\mathbf{I}_n)U_n,U_n\right\rangle\right]$$
$$+E[\left\langle(\mathbf{K}_n+\mathbf{D}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{D}_n+\mathbf{F}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{F}_n+\varepsilon\mathbf{I}_n)W_nX_n,W_nX_n\right\rangle$$
$$-2E\left[\left\langle(\mathbf{K}_n+\mathbf{D}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{D}_n+\mathbf{F}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{F}_n+\varepsilon\mathbf{I}_n)W_nX_n,U_n\right\rangle\right]$$
$$+E\left[\left\langle(\mathbf{K}_n+\mathbf{D}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{D}_n+\mathbf{F}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{F}_n+\varepsilon\mathbf{I}_n)U_n,U_n\right\rangle\right]$$
$$=E\left[\left\langle R^{\varepsilon}_{n}X_{n},X_{n}\right\rangle-\left\langle \mathbf{C}^{*}\mathbf{C}X_{n},X_{n}\right\rangle-\left\langle(\mathbf{K}_n+\varepsilon\mathbf{I}_n)U_n,U_n\right\rangle\right]$$
$$+E\left\langle(\mathbf{K}_n+\mathbf{D}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{D}_n+\mathbf{F}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{F}_n+\varepsilon\mathbf{I}_n)(W_nX_n-U_n),(W_nX_n-U_n)\right\rangle.$$

for all $n=0,..,N-2$. Summing for $n=0$ to $N-2$ the last equality, we obtain

$$E\left[\left\langle R^{\varepsilon}_{N-1}X_{N-1},X_{N-1}\right\rangle\right]=E\left[\left\langle R^{\varepsilon}_{0}X_{0},X_{0}\right\rangle\right]-$$
$$\sum_{k=0}^{N-2}E\left[\left\langle \mathbf{C}^{*}\mathbf{C}X_k,X_k\right\rangle\right]+E\left[\left\langle(\mathbf{K}_k+\varepsilon\mathbf{I}_k)U_k,U_k\right\rangle\right]$$
$$+\sum_{k=0}^{N-2}E\left[\left\|(\mathbf{K}_n+\mathbf{D}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{D}_n+\mathbf{F}^{*}_{n}R^{\varepsilon}_{n+1}\mathbf{F}_n+\varepsilon\mathbf{I}_n)^{1/2}(W_nX_n-U_n)\right\|^2\right].$$

On the other hand, from (4.1), the definition of K^{ε}_{N-1} and (4.2), we get

$$E\left[\left\langle R^{\varepsilon}_{N-1}X_{N-1},X_{N-1}\right\rangle\right]=$$
$$E[\left\langle\left(\mathbf{C}^{*}\mathbf{C}+\mathbf{A}^{*}\mathbf{S}\mathbf{A}+\mathbf{B}^{*}\mathbf{S}\mathbf{B}\right)X_{N-1},X_{N-1}\right\rangle$$
$$-\left\langle K^{\varepsilon}_{N-1}W_{N-1}X_{N-1},W_{N-1}X_{N-1}\right\rangle]$$
$$=E[\left\langle\left(\mathbf{C}^{*}\mathbf{C}+\mathbf{A}^{*}\mathbf{S}\mathbf{A}+\mathbf{B}^{*}\mathbf{S}\mathbf{B}\right)X_{N-1},X_{N-1}\right\rangle$$
$$-\left\langle K^{\varepsilon}_{N-1}(W_{N-1}X_{N-1}-U_{N-1}),(W_{N-1}X_{N-1}-U_{N-1})\right\rangle$$
$$+2\left\langle\left(\mathbf{D}^{*}_{N-1}\mathbf{S}\mathbf{A}+\mathbf{F}^{*}_{N-1}\mathbf{S}\mathbf{B}\right)X_{N-1},U_{N-1}\right\rangle+\left\langle K^{\varepsilon}_{N-1}U_{N-1},U_{N-1}\right\rangle].$$

Combining the last two formulas, we obtain the conclusion of the Lemma.

5. Main results

In this section we shall prove that problem $\mathcal{O}$ has a solution derived from the solution of problem $\mathbf{O}_{\varepsilon}$.

Proposition 1. For all $\varepsilon>0$
$$\min_{U\in \mathbf{U}^{a}_{0,N-1}} I_{x_0,N,\varepsilon}(U)=\min_{u\in \mathbf{\mho}^{a}_{0,N-1}} I_{x_0,N}(u).$$

Proof. Let $u=\{u_0,..,u_{N-1}\}\in \mathbf{\mho}^{a}_{0,N-1}$. If $U=\{U_0,..,U_{N-1}\},U_k=\left(0,..,\underset{k}{u_k},...0\right)$, then $U\in \mathbf{U}^{a}_{0,N-1}$ and $I_{x_0,N,\varepsilon}(U)=I_{x_0,N}(u)$. Thus

$$\min_{u\in \mathbf{\mho}^{a}_{0,N-1}} I_{x_0,N}(u)\geq \min_{U\in U} I_{x_0,N,\varepsilon}(U). \qquad (5.1)$$

On the other hand if $U\in \mathbf{U}^{a}_{0,N-1}$ and $u=\{u_0,..,u_{N-1}\}$ is defined by $u_k=U_{kk},k=0,1,..,N-1,$ then $u\in \mathbf{\mho}^{a}_{0,N-1}$ and $I_{x_0,N}(u)=I_{x_0,N}(U)\leq I_{x_0,N,\varepsilon}(U)$. Replacing U from the above inequality by $\overline{U}^{\varepsilon}$, the optimal control which minimizes $I_{x_0,N,\varepsilon}(U)$ (we know that it exists), we see that $I_{x_0,N,\varepsilon}(\overline{u})\leq I_{x_0,N}(\overline{U}^{\varepsilon})=\min_{u\in \mathbf{\mho}^{a}_{0,N-1}} I_{x_0,N}(\overline{U}^{\varepsilon})$,
where $\overline{u}=\{\overline{u}_0,..,\overline{u}_{N-1}\}$ and $\overline{u}_k=\overline{U}_{kk}$. Therefore
$\min_{u\in \mathbf{\mho}^{a}_{0,N-1}} I_{x_0,N}(u)\leq \min_{U\in \mathbf{U}^{a}_{0,N-1}} I_{x_0,N,\varepsilon}(U).$
In view of (5.1) we get the conclusion.

The next theorem is a direct consequence of Lemma 1 and of the above proposition.

Theorem 1. The unique solution $\{R^{\varepsilon}_{n}\}_{n=0,..,N-1}$ of the Riccati equation (4.1) does not depend on ε and $\min_{u\in \mathbf{\mho}^{a}_{0,N-1}} I_{x_0,N}(u)=E\left[\left\langle R^{\varepsilon}_{0}X_0,X_0\right\rangle\right].$

Proof. Let $W_n,n=0,..,N-1$ be defined by (4.2), (4.3). The control sequence $\overline{U}=\{\overline{U}_0=W_0X_0,..,\overline{U}_n=W_nX_n,....,$ $\overline{U}_{N-1}=W_{N-1}X_{N-1}\}$
minimizes the cost functional $I_{x_0,N,\varepsilon}(U)$ and
$\min_{U\in \mathbf{U}^{a}_{0,N-1}} I_{x_0,N,\varepsilon}(U)=E\left[\left\langle R^{\varepsilon}_{0}X_0,X_0\right\rangle\right]$.
Moreover, the control $\overline{u}=\{\overline{u}_0,..,\overline{u}_{N-1}\}$ defined by $\overline{u}_k=\overline{U}_{kk}$ is also optimal for

$I_{x_0,N}(u)$ and $\min_{u \in \mathcal{U}^a_{0,N-1}} I_{x_0,N}(u) = E\left[\left\langle R_0^{\varepsilon} X_0, X_0 \right\rangle\right]$

Since $\min_{u \in \mathcal{U}^a_{0,N-1}} I_{x_0,N}(u)$ does not depend on ε it follows that R_0^{ε} does not depend on ε.

Reasoning as above for a modified initial time n_0 of the control system we deduce that $R_{n_0}^{\varepsilon}$ does not depend on ε. Inductively, we deduce that $\{R_n^{\varepsilon}\}_{n=0,..,N-1}$ has a similar property. The proof is complete.

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